Research proposal: moduli problems in skeletal and tropical geometry

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Background

Non-Archimedean geometry has, since its genesis in the 60s, become a substantial research topic in arithmetic geometry, Arakelov theory, resolutions of singularities, and increasingly, Kähler geometry. It provides a geometric language to unify the study of two disparate classes of objects:

- p-adic analytic spaces;

- formal degenerations of complex manifolds.

It therefore acts as a bilingual dictionary that relates transcendental features of arithmetic to complex analysis, much as Grothendieck's scheme theory operates for algebraic features.

The theory of skeletons in non-Archimedean geometry is a new subject that aims to capture information about these spaces in terms of combinatorial and integer-affine structures. Beginning with the work of Berkovich [Ber99] - in which a skeleton is treated as a certain polyhedral complex - it has found application as a unifying framework that incorporates also tropical geometry [BPR11] and a version of the SYZ phenomenon in mirror symmetry [KS06]. In each of these latter areas, a critical role is played by the fact that the skeleton carries more structure than that considered by Berkovich.

Broadly speaking, the programme of skeleton theory is as follows. Starting with an analytic space X, one tries to construct a *collapse map*

$$\mu : X \to B$$

onto a *skeleton* B. A general fibre of this collapse map is a 'purely imaginary torus' - that is, a torsor for a unitary group $(\mathrm{U}_1)^{\dim X}$. Over the locus of torus fibres, one can make sense of the notion of *affine function* on B. The local structure of this region is simple enough that the study of X reduces to the study of B and the special - or 'singular' - fibres of μ.

Attempting to carry out this programme can be divided neatly into two stages:

i) construct a 'good' skeleton B with a collapse map $X \to B$;

ii) use the geometry of B to deduce information about X.

The kinds of invariants we may wish to compute include:

- cohomology of X (de Rham, étale, cyclic, etc.);
- the derived category of X;
- the deformation space of X.

For example, in the singularity-free setting of completely degenerate analytic tori over a field $\mathbb{C}((t))$ of formal Laurent series, one can choose B to be an *affine manifold*, whereupon the de Rham cohomology of X is

$$H^{p,q}_{\mathrm{dR}}(X/\mathbb{C}((t))) = H^q(B; \wedge^p \Lambda \otimes \mathbb{C}((t)))$$

where $\Lambda \subset T^\vee B$ is the local system of 'integer' one-forms, which are locally the differentials of affine functions. The scope of such calculations is more general - for instance, Gross and Siebert [GS07] have obtained a generalisation to certain Calabi-Yau degenerations with simple singularities.

Minimal skeletons Let us choose a model X^+ of X - that is, a choice of special fibre for the degeneration. The basic example of a skeleton of X is the dual intersection complex $\Delta(X^+)$, whose cells parametrise strata of this central fibre.

There are many models of X, but the choice may be cut down substantially by running a Minimal Model Programme (MMP). The dual intersection complex of a minimal model is a *minimal skeleton*. Like minimal models, minimal skeletons are not, in general, unique; however, unlike arbitrary skeletons they should depend only on finitely many parameters. A minimal skeleton also has a better-behaved structure than a general one - for instance, if X is Calabi-Yau, it is expected to be an orientable manifold.

In positive and mixed characteristic, the unavailability of resolutions of singularities or the MMP makes it difficult to guarantee the existence of skeletons - especially minimal ones - coming from geometry in this manner. For number-theoretic applications (like the computation of p-adic integrals) one therefore hopes for a 'purely skeletal' construction.

Lifting problem Suppose that we have understood stage *i)* well enough to define a parameter space of pairs consisting of an analytic space and a 'good' (e.g. minimal) skeleton. By forgetting the analytic space, there is then a natural *modular collapse* map

$$\{\text{analytic space} + \text{skeleton}\} \xrightarrow{\mu} \{\text{skeleton}\} \tag{1}$$

from this space to a parameter space of abstract skeletons. The fibre over a skeleton B is the space of *lifts* of B. The study of this *lifting* or *reconstruction* problem is the central focus of tropical geometry and the Gross-Siebert programme.

In the degenerations appearing in the Gross-Siebert programme, the space of lifts is a torus. It is therefore natural to ask for a statement of the form:

The moduli space of B is a skeleton of the moduli space of X.

For instance, Abramovich, Caporaso, and Payne have formulated and proved a combinatorial result of this kind that relates the moduli space of curves to a certain moduli space of closed graphs [ACP15].

Objectives

In this project, I plan to apply the algebraic foundations of skeleton theory, which I began in my thesis [Mac14], to the study of certain *moduli problems*. The salient classes of moduli problems will be the following three:

i) skeletons of a fixed analytic space; *minimal skeleton problem*

ii) analytic spaces with fixed skeleton; *lifting problem*

iii) pairs consisting of an analytic space and a (minimal) skeleton.

Part of the programme will be to find precise definitions of these moduli problems as functors.

Skeletal Kähler-Ricci flow With existing technology, it is already possible to make precise the notion of moduli space $\mathscr{S}k(X)$ of skeletons of a fixed analytic space X. We will regard $\mathscr{S}k(X)$ as a kind of polyhedral complex. It is infinite-dimensional, but by controlling the 'size' of the skeleton, it can be filtered by finite-dimensional subcomplexes.

The smallest, and most important, stage of this filtration is the locus of minimal skeletons. In this paragraph, I will propose a way to use the minimal model programme to relate larger strata of $\mathscr{S}k(X)$ to this locus.

In explicit examples - such as a non-singular blow-down - it is possible to 'interpolate' a stage of the MMP for a model X^+ of X as a *family* of skeletons indexed by an interval, that is, a retract of $X\times[0,1]$ over $[0,1]$. Running an MMP therefore determines a path

$$[0,n]\to\mathscr{S}k(X) \tag{2}$$

that takes 0 to the dual complex of X^+ and converges, as $n\to\infty$, into the locus of minimal skeletons. In particular, it defines a homotopy equivalence between $\Delta(X^+)$ and a minimal skeleton (if there are no flips, it is even a deformation retract).

We would like to find a path (2) that works consistently for the entire space of skeletons, inducing a deformation retract onto the minimal locus. However, the MMP depends on choices at each stage, and can terminate with different minimal models.

One way to eliminate these choices is to fix a polarisation L of X and run the MMP for a sufficiently generic pair $(X^+;L^+)$ modelling $(X;L)$. The space of extensions of the polarisation over a model is a finite-dimensional approximation to the space of *semipositive non-Archimedean metrics* on L, which forms a family of positive cones

$$\mathrm{Met}^+(X;L)\to\mathscr{S}k(X)$$

inside an infinite-dimensional vector bundle, modelled on the space of piecewise-linear functions on X, over $\mathscr{S}k(X)$.

The idea is then to interpolate the MMP *algorithm* as a kind of 'flow'

$$[0,\infty)\to\mathrm{End}(\mathrm{Met}^+(X;L))$$

on the bundle of semipositive metrics. This flow is an analogue of the Kähler-Ricci flow in differential geometry. At least in the Calabi-Yau case, it should be characterised by convergence to a solution of a *real Monge-Ampère equation*, which is predicted by mirror symmetry.[1]

Since the minimal model programme for surfaces, even in positive and mixed characteristic, is straightforward, it should be possible to understand this flow explicitly in the case of degenerating curves. In particular, the minimal skeleton is unique (for $g > 0$), and so the Kähler-Ricci flow should define a contraction of $\mathscr{S}k(X)$.

Objective. *Define and study convergence of the skeletal Kähler-Ricci flow, beginning with the case of curves.*

Abelian varieties When X is a completely degenerate Abelian variety over, say, a discrete valuation field K, a famous construction of Mumford [Mum72] gives us a unique minimal skeleton, allowing us to bypass the minimal model programme. The skeleton in this case is a real torus with a complete affine structure. This singularity-free situation is an essentially trivial generalisation of toric geometry.

The modular collapse map (1) therefore goes

$$\mu : \{K\text{-analytic tori}\} \to \{\mathbb{Z}\text{-affine tori}\}$$

from the set of degenerate K-analytic tori to the set of real tori with $\mathbb{Z}$-affine structure. The fibre over a fixed torus B is a torsor for $H^1(B;\mathrm{Hom}(\Lambda,\mathrm{U}_1(K))$; in other words, μ is also a non-singular torus fibration. This observation opens the door to explicit information about the cohomology 'near the cusp' of the moduli space of Abelian varieties.

With the theory of rigid analytic spaces over $\mathbb{F}_1((t))$ [Mac15a, Mac15b], it is possible to refine this intuition to a precise statement about moduli functors.

Objective. *Develop the theory of non-singular non-Archimedean torus fibrations to the point at which we may formulate and prove these statements.*

K3 surfaces When X is a K3 surface, the lack of a purely toric degeneration entails some interesting complications:

- the components of the central fibre are not toric varieties, but more general *log Calabi-Yau* surfaces;
- different minimal models of X over $\mathcal{O}_K$ in general determine different minimal skeletons; the 'minimal locus' of $\mathscr{S}k(X)$ has positive dimension.

The first issue means that the *local* lifting problem is more subtle than simple toric geometry. However, with the recent Torelli theorem for log CY surfaces [GHK12], it is now entirely accessible. The space of global lifts should again be a torsor for a first cohomology group $H^1(B;\mathrm{Hom}(\Lambda^{\mathrm{mod}},\mathrm{U}_1))$ with coefficients in the dual of some constructible extension Λ^{mod} of the local system Λ on B.[2]

[1] Other authors [BFJ11] have already studied a related flow on a space of semipositive metrics on X that does not keep track of a skeleton, obtaining a solution to a 'non-Archimedean Monge-Ampère equation'; this is likely closely related to my proposal.

[2] The aforementioned authors are also working on a globalisation in the context of their programme, though at time of writing this paper has yet to appear.

The minimal locus of the moduli space $\mathscr{S}k(X)$ of skeletons of X is a parameter space of singular affine structures on the 2-sphere. The space of *all* affine structures on a 2-sphere is a 'skeletal' analogue of the period domain of polarised K3 surfaces.

Objective. *Construct the skeletal K3 period domain - that is, the moduli space of affine structures on the 2-sphere - and investigate its relationship with the classical one.*

Much of the technology needed to achieve this objective is already in place.

Finally, although the process of extracting a minimal skeleton for K3 cannot usually be bypassed, the study of so-called 'Type III' K3 degenerations is classical and hence may already be sufficiently well-understood to study explicitly the skeletal Kähler-Ricci flow in this case. The fixed points of this dynamical system should give non-trivial examples of global solutions to real Monge-Ampère equations in on 2-spheres.

Objective. *Investigate the skeletal Kähler-Ricci flow for K3 degenerations.*

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