Affine manifolds are rigid analytic spaces in characteristic one

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Abstract

I extend the definitions of schemes relative to monoids with zero - and therefore, toric geometry - to the world of formal schemes and rigid analytic spaces. This expands the usual framework to include, for instance, models for Mumford's degenerating Abelian varieties. I also reformulate the traditional notions of separated and proper morphism in a manner amenable to the context of relative geometry.

The category of affine manifolds embeds into the category of such rigid analytic spaces as a subcategory defined by simple algebraic (normal) and topological (overconvergent) criteria. The affine manifold of a rigid space can be recovered either as a set of 'Novikov field' points or as a universal Hausdorff quotient. After base change to any topological field, one obtains a 'toric' analytic space that fibres over the affine manifold.

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Part I

Formal geometry

1 Introduction

Toric geometry was introduced, in the wake of the pioneering work [GK73], as an organising principle for the ubiquitous appearance of monomial techniques in algebraic geometry. Abstractly, it can be viewed as an attempt to answer the question

Which algebraic varieties can be defined without addition? (1)

The absence of addition - or equivalently, the large torus symmetry - makes understanding such varieties substantially simpler as compared to more general varieties. Indeed, toric varieties famously can be captured entirely in terms of a combinatorial object: a *fan* in a rational vector space. For this reason, they have provided a fertile testing ground for theories in algebraic geometry that are too difficult to tackle in more general situations.

In this paper, I address the following generalisation of question 1:

Which analytic spaces can be defined without addition? (2)

The analogue to the answer that was provided before by toric geometry is here given by analytic spaces with the structure of a *maximal torus fibration* over an affine manifold. The affine manifold adopts the rôle of parametrising object that in algebraic geometry was played by the fan.

While the hypothesis of being defined by monomials is still very restrictive locally, the parametrisation by affine manifolds shows that passing to the analytic setting makes available much more interesting *global* geometry. For instance, part of Mumford's theory of completely degenerate Abelian varieties [Mum72] can be described in this way; skip to §12.7 for a discussion. Still, toric analytic spaces represent a substantial simplification of the theory of analytic spaces in general.

The correspondence between maximal torus fibrations over $\mathbb{C}$ and their parametrising affine manifolds has an elementary geometric description: by definition, an affine manifold B carries a certain local system Λ_B of *integer 1-forms*, and the corresponding complex analytic space is defined to be the torus bundle

$$\pi : TB/\epsilon\Lambda_B^\vee \to B$$

depending on a parameter ϵ, whose monomial holomorphic co-ordinates $\exp\epsilon(\pi^*F + 2\pi i dF)$ are indexed by affine functions F on B with integer differential.

The focus of this paper is rather in the rigid analytic, or 'non-Archimedean', paradigm. In other words, the basic object of study is a *formal degeneration* - which already appeared in the previous formula in the form of the parameter ϵ. From this perspective, the geometric objects that become available are particularly degenerate formal families that not only have toroidal crossings in the sense studied in [GK73], but all of the components of whose central fibre are toric varieties. Such families arise as the e^ϵ-adic completion of the corresponding complex-analytic objects.

Of course, these ideas have also found plenty of attention in the literature - indeed, the basic picture is already implicit in the work of Mumford. A particular source of inspiration for me has been the Gross-Siebert programme in mirror symmetry [Gro11] and the paper [KS06], in which the authors define a notion of 'non-Archimedean torus fibration' which was the starting point for the present work.

Characteristic one

A more fundamental approach to question (1) is to try to define *algebraic geometry itself* without addition. This is the method of 'relative' algebraic geometry (cf. [TV09]). While toric geometry has the advantage of being very *concrete*, relative geometry is heavily modelled on the abstract framework of traditional algebraic geometry and so is very *robust*. It is possible to push through much of the basic structure of [Gro60] in a very general setting (see, for example, [Dur07]).

Under this approach, one is free to replace the defining algebraic objects of geometry over $\mathbb{Z}$ - the commutative rings - with a commutative algebraic object of one's choosing. Since we are interested in geometry without addition, the obvious choice here is to work with *multiplicative monoids*. General principles then provide us with a category of 'schemes' equipped with a structure sheaf of monoids.

The essential features of any object defined in such a geometry are necessarily independent of 'coefficients' - for instance, considerations of characteristic. For this reason, algebraic geometry relative to monoids formed the basis of some early attempts to make sense of geometry relative to the mysterious object $\mathbb{F}_1$, the so-called 'prime field of characteristic one' ([Dei08], [CC09]). It is this connection that gives the present paper - and our category of schemes relative to monoids - its name.[1]

Apart from being more structured than toric geometry, geometry in characteristic one is also more general - but not *too much* more general, as the following basic result shows:

Theorem ([Dei08])**.** *Let X be a normal, connected $\mathbb{F}_1$-scheme, separated and of finite type over $\mathbb{F}_1$. The base change of X to $\mathbb{Z}$ is a toric variety.*[2]

In other words, under some natural - and rather light - hypotheses of an algebro-geometric nature, $\mathbb{F}_1$-geometry can be understood using the machinery of cones and fans. A large part of this work is devoted to developing a more precise version of this statement and an analogue in the analytic setting.

Cone complexes

In part I, I address the case of formal geometry. Although we make no discussion of analytic spaces or affine manifolds until part II, the nature of the definition will be such that many of the salient features already appear at the formal level.

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[1] I should point out that arithmeticians quickly realised that monoids were completely inadequate for questions of arithmetic, and turned to consider more sophisticated objects [Bor09, Dei13, Dur07, Lor12]. Since geometry, rather than arithmetic, is the primary motivation for this paper, beyond a few scattered comments I do not mount any attempt to study such ideas. See remark 2.5.

[2] Strictly speaking, the notion of *separated* $\mathbb{F}_1$-scheme has not, as far as I know, been defined in the literature. In this paper, I address this deficiency (§4).

Before introducing the objects that 'parametrise' toric formal schemes, I would like to present a more precise formulation of Deitmar's theorem:

Theorem (6.1)**.** *The category of connected, normal, separated, finite type $\mathbb{F}_1$-schemes and non-boundary morphisms is equivalent to the category of rational polyhedral fans.*

The hypotheses appearing in this statement can be divided into three groups. The first - normal and non-boundary - is essential to get any kind of description in terms of cones. Here 'non-boundary' essentially means that the morphisms are restricted not to land in any closed subscheme. The closed subschemes are the part of an $\mathbb{F}_1$-scheme that becomes the toric boundary upon base change to $\mathbb{Z}$.

The second group consists of the 'finite type' assumption. It is easy to alleviate this hypothesis, should we desire, by considering more general kinds of cone.

The third group of assumptions are of a topological nature, and are therefore almost completely invisible from the usual algebro-geometric standpoint. To help understand them, we will increase generality, and ask: what do *completely general* normal, locally finite type $\mathbb{F}_1$-schemes look like? The traditional picture has already given us to understand *affine* objects: they are the fans consisting of a single cone embedded in a vector space N. Since each normal, locally finite type $\mathbb{F}_1$-scheme X is glued together from affine pieces, a global object can be understood in terms of a certain *cone complex* Σ_X, which consists of a collection of embedded cones $\sigma \subset N$ glued together along faces in a way that respects the embedding in N.

Now we can ask what our topological hypotheses look like on cone complexes. The connectedness assumption is somewhat obvious, so we may restrict to the connected case. Now observe that since every connected, integral scheme is contractible onto its generic point, there is no topological obstruction to patching together the embeddings of constituent cones into N. We therefore get a globally defined, locally linear map

$$\delta : \Sigma_X \to N_X$$

which, borrowing terminology from the manifolds literature, we may call the *developing map*. By unwinding the definitions, it is not difficult to see:

$$X \text{ separated} \qquad \Leftrightarrow \qquad \delta \text{ injective}.$$

In real life, one usually wishes to restrict attention to separated objects, and so this result renders the abstraction of cone complexes somewhat pointless for algebraic geometry.

The analogue of the above statement is *false* in formal geometry. Indeed, it is no longer true that integral formal schemes are 0-types, and so δ need not even have a global extension. The theory of cone complexes is therefore unavoidable in formal geometry.

1.1 Summary of results

The main results of this part give combinatorial characterisations of separated and proper morphisms of formal $\mathbb{F}_1$-schemes. However, the literature has apparently so far omitted to provide definitions of these terms, other than to remark that the definitions of separated and proper morphisms given in [Gro60] clearly do not work 'out of the box'. Therefore, before stating the results, I must clarify what I am talking about.

My approach to defining propriety in this paper has been in the spirit of generalising the valuative criterion. Rather than giving the general definition here, I provide a list of equivalent characterisations in cases where the criteria can be couched in more familiar terms (but see §1.2):

Theorem. *Let S be a formal scheme over $\mathbb{Z}$ or $\mathbb{F}_1$. Let $f : X \to S$ be of finite type. The following conditions are equivalent:*

i) ($S/\mathbb{Z}$) f is separated and universally closed (Thm. 4.39);

ii) (S qcqs and admits an ample bundle) for every open subset $U \subseteq X$ quasi-projective over S, there exists a Chow cover

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "\exists\mathbb{P}"] \arrow[dr, bend left, "\mathbb{P}"] & \\
U \arrow[ur, bend left, "\exists"] \arrow[r, hook, "\forall\circ"] \arrow[drr, bend right, "quasi\text{-}\mathbb{P}"'] & X \arrow[dr] & \\
 & & S
\end{tikzcd}$$

projective over S and admitting a section over U (Thm. 4.45);

iii) (S Noetherian) f satisfies the valuative criterion for propriety (Cor. 4.48).

For case iii)*, it is enough that every embedded subscheme of X surjects onto the embedded closure of its image in S, and that this remains true after any base change (Thm. 4.53).*[3]

Note that the Chow property *ii)* stated here is actually much stronger than the usual statement of the Chow lemma, which only asserts the *existence* of an open dense U. It is this property that follows immediately from our definition of propriety. The challenge lies in establishing sufficient conditions for checking it in practice.

With rigid geometry in mind, it is actually natural to give more attention to *overconvergent* morphisms. Intuitively, these are morphisms that satisfy the valuative criterion without necessarily being quasi-compact. This class of morphisms is not visible in ordinary toric geometry, since a connected integral scheme can be overconvergent only if it is proper. However, they are essential for understanding the rigid analytic world considered in part II.

A large part of this paper (§4) is devoted to developing this machinery. See §1.2 below for a summary of the approach.

An affine formal scheme is always obtained as the formal completion of a scheme. Topologically, the effect of formal completion is to remove certain open sets. On the combinatorial side, we must therefore respond by removing certain faces of the corresponding cone. To every affine, normal formal $\mathbb{F}_1$-scheme we may therefore associate an object in a category $\mathbf{Cone}_*^N$ of *punctured cones* embedded in a rational vector space $N(\mathbb{Q})$.

Globalising, we obtain also a category $\mathrm{C}\mathbf{Cone}_*^N$ of *punctured cone complexes* and a *cone complex functor*

$$\Sigma : \mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{n/nb}} \to \mathrm{C}\mathbf{Cone}_*^N$$

---

[3] Note that over $\mathbb{F}_1$, 'embedded' means something more general than 'closed embedded'; see §3.1.

from the category of normal formal $\mathbb{F}_1$-schemes with non-boundary morphisms. From general principles a classification - stated here, for simplicity, under additional finiteness assumptions - follows:

Theorem (7.10)**.** *The cone complex functor restricts to an equivalence of categories between the category of normal, locally Noetherian formal $\mathbb{F}_1$-schemes with non-boundary morphisms and the category of integral polyhedral punctured cone complexes.*[4]

As indicated above, any punctured cone complex Σ_X comes equipped with a locally defined developing map $\delta : \Sigma_X \to N_X$. The primary results on cone complexes are stated in terms of this map:

Theorem (7.21)**.** *Let X be a locally Noetherian, integral formal $\mathbb{F}_1$-scheme. Then X is separated if and only if the developing map of $\Sigma_X(\mathbb{R})$ is a local immersion.*

Theorem (7.24)**.** *Let $f : X \to S$ be a non-boundary morphism between locally Noetherian, integral formal $\mathbb{F}_1$-schemes. Suppose that f is paracompact and locally of finite type.*

Then f is overconvergent if and only if $\Sigma_X(\mathbb{R}) \to \Sigma_S(\mathbb{R})$ is a topological submersion. In particular, in this case its fibres are manifolds without boundary.

If f is overconvergent, the developing map equips the fibres of the induced map of cone complexes with a canonical structure of a *radiant affine manifold* (cf. [GH84a]) with an action of $\mathbb{R}^\times_{>0}$. The affine manifold promised in the title and described in part II will be the quotient by this action.

Finally, and somewhat incidentally to the rest of the development, the intuition provided by cone complexes permits us to prove an algebraisation criterion:

Theorem (7.27)**.** *Let X be an integral formal scheme over $\mathbb{F}_1$. The multiplicative group $K^\times_X$ of the function field is a locally constant sheaf.*

Suppose that X is connected; then it is algebraisable if and only if $K^\times_X$ is constant. In this case, the algebraisation can be made functorial.

Of course, algebraic $\mathbb{F}_1$-schemes being toric varieties, algebraisability over $\mathbb{F}_1$ is a much stronger hypothesis than algebraisability of a base change to $\mathbb{Z}$. Such examples are completely uninteresting from the perspective of affine manifold theory.

1.2 Structural features

Schemes relative to monoids have already found attention from various perspectives in the literature. Much of sections 2, 3, and 5 are devoted to further developing the basic properties of schemes and formal schemes to bring them more into line with the early parts of [Gro60]. Since these notions behave, with mild deviations, much like their counterparts over $\mathbb{Z}$, I relegate any summary of these sections to the body of the text.

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[4] A tweak of the definition of Σ allows this statement and the two below to work for formal schemes locally of finite type over a possibly non-Noetherian rank one valuation $\mathbb{F}_1$-algebra.

Criteria for separation, propriety, overconvergence The notions of separated and proper morphisms are traditionally defined in terms of *closed maps*. As some authors [Dur07, §6.5.20] have already noted, in algebraic geometry relative to monoids there aren't enough closed sets for these definitions to produce something meaningful.

The generalisation from schemes to rigid analytic spaces, too, diminishes the relevance of closed subsets to geometry, though the rigid geometry community have nonetheless managed to push something through along classical lines [FK13, II.7.5].

In [Dur07, §6.5.23], it is suggested that one could approach this problem by generalising the notion of closed subscheme to *embedded* subschemes - that is, subschemes defined by equations (cf. §3.1). Although we discuss that approach, I have chosen to focus instead on adapting the *valuative criteria*. The only unsatisfactory aspect of the valuative criteria as they stand is their reliance on valuation rings.

So what we are looking for is a notion of valuative criterion *without the valuation*; that is, the problem of extending a morphism $U \to X$ (over a base S) along a *completely arbitrary* open immersion $U \subseteq V$:

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & V \arrow[d] \arrow[dl, dashed, "?"'] \\
X \arrow[r] & S
\end{tikzcd}$$

Of course, it's unrealistic to expect that such an extension could exist without first modifying V - the only reason we got away with it in the classical valuative criterion is that every finitely generated ideal in a valuation ring is principal, and hence cannot be blown up.

The geometric input to the theory lies in specifying exactly what kinds of modifications are allowed. My choice of the word 'modification' is no coincidence: for algebraic schemes I will allow any scheme *projective* over V admitting a section over U. Such a V-scheme is called an *overconvergent neighbourhood* of U/V.

So for overconvergence of X/S, we are looking for an extension

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & \tilde{V} \arrow[r, "\mathbb{P}"] \arrow[dl, dashed, "!"'] & V \arrow[d] \\
X \arrow[rr] & & S
\end{tikzcd}$$

of $U \to X$ to an overconvergent neighbourhood $\tilde{V}$ of U/V, and this extension should be unique up to refinement - that is, further modification - of $\tilde{V}$. A morphism is proper, by definition, if it is overconvergent, quasi-compact, and quasi-separated.

The class of proper morphisms resulting from this definition turns out to be exactly those morphisms satisfying a strong version of the Chow lemma locally on the base (thm. 4.45). The proof of this lemma for finite type, separated, and universally closed morphisms of schemes over $\mathbb{Z}$ - and hence that this theory recovers the usual one as presented in [Gro60] - relies on flattening stratification. An analogous statement for $\mathbb{F}_1$ seems quite distant at the moment, and so our definition is, at least *a priori*, rather more powerful than any definition based around the notion of embedded subscheme - but see §4.8.5.

Another benefit to this kind of approach is that most of the basic structural results are 'soft', meaning that they can be abstracted. We can therefore systematically define separated, proper, and overconvergent morphisms in any of the menagerie of topoi considered in this paper, the only input being a class of morphisms $\mathbb{P}$ satisfying a short list of conditions.

This allows us to mount a comparison of these notions in contexts related by morphisms of topoi. The comparison principles outlined in §4.3 are of a somewhat technical nature; however, in practice they tend to rely on genuinely geometric input. The conclusions of the theory for formal $\mathbb{F}_1$-schemes are:

- overconvergence of morphisms locally of finite type between formal schemes can be checked using finite type reduced schemes as test spaces (§4.4);
- if the the source is integral or the base is Noetherian, one can check using only *normal* test spaces (§5);
- in the latter case, one can even use only the specific test space $V = \mathbb{A}^1_{\mathbb{F}_1} = \operatorname{Spec}\mathbb{F}_1[t]$.

The arguments for the first item are valid over $\mathbb{Z}$, and those for the second can most likely be extended with minor modifications; since the conclusions are anyway well-known in that setting, I didn't try to confirm this.

Capturing geometry in terms of combinatorial data The constructions of combinatorial gadgets for $\mathbb{F}_1$-schemes, formal schemes, and rigid analytic spaces all follow the same general programme. I take a moment here to summarise the salient features. The initial steps are:

i) assignment of combinatorial object to affine scheme;

ii) description of open sets;

iii) when does the combinatorial object determine the affine object?

For example, in the case of finite type $\mathbb{F}_1$-schemes - approximately, toric varieties - these initial steps go:

- an affine toric variety gives rise to a *rational polyhedral cone* in a vector space with a lattice;
- open sets correspond to *faces* of the cone;
- the cone determines the variety precisely when it is *normal*.

This description becomes more useful when we have also:

iv) description of points;

v) topological realisation.

In toric geometry, the integer points are usually interpreted as one-parameter subgroups of the embedded torus. More generally, the H-points, for general additive subgroups $H \subseteq \mathbb{R}$, have the interpretation of certain rank one valuations. Over $\mathbb{F}_1$, the notion of a valuation is essentially equivalent to that of a non-boundary *jet*, that is, a point valued in the spectrum of a valuation $\mathbb{F}_1$-algebra with value group H.

In the special case $H = \mathbb{R}$, one can use the order topology to obtain a *topological realisation* $\sigma(\mathbb{R})$ of a combinatorial object σ. In general, this will be some kind of convex, semi-linear set inside a real vector (or in part II, affine) space.

The affine picture well-understood, the next stage is globalisation:

vi) glueing.

This is an exercise in abstract nonsense: the language of locally representable sheaves (cf. §1.3) reduces the question of globalising a functorial construction to that of checking a couple of basic categorical properties, to whit,

vi*a*) is it flat?[5]

vi*b*) does it preserve open immersions?

vi*c*) does it preserve coverings?

The first item does not, for the most part, present a serious difficulty (but see the proof of theorem 7.27). The second criterion will follow automatically from our understanding of item ii), and the third does not even appear until part II.

The topological realisations of constituents glue together to give a compactly generated Hausdorff space $\Sigma_X(\mathbb{R})$ whose topological properties reflect those of X. For instance, a locally Noetherian scheme X is quasi-compact if and only if $\Sigma_X(\mathbb{R}) \setminus 0$ is conically compact, that is, compact up to the action of $\mathbb{R}^\times_{>0}$.

More interesting conditions on X manifest in the study of the

vii) developing map, which we use to provide

viii) overconvergence criteria.

We already concluded that overconvergence for normal schemes is detected by normal test spaces, and so it follows that it can equally be calculated in the category of cone complexes.

Thus once we understand how *modifications* look on the category of cone complexes, we can understand criteria for separation, propriety, and overconvergence directly from the definition via extension problems.

Returning to the example of classical toric geometry, we obtain a rephrasing of some well-known facts:

- to an algebraic $\mathbb{F}_1$-scheme we associate a locally representable presheaf on the category of cones - in other words, a complex of cones, glued together along faces, such that no two faces of the same cone are glued together;
- the developing map is defined globally on each connected component;
- a birational modification is a subdivision;
- the separation criterion is that δ is an *immersion*, and therefore gives rise to a *fan* in a rational vector space;

---

[5] This is a fancy way of saying 'would be left exact, if the category were finitely complete'. We need this for a stupid reason: our categories of combinatorial data do not contain an object representing the empty set!

- the overconvergence criterion is that δ is a *homeomorphism*; that is, the fan has support the whole space.

The real crux is that for formal schemes, the developing map is not globally defined, and so the criteria for separation and overconvergence have to be replaced with *local* conditions. This is the punchline of part I (§7.5).

1.3 Regarding topoi

In this paper, the notion of a geometry admitting an *atlas* by some specified class of objects is formalised by the concept of *locally representable sheaf*. The context in which this makes sense is captured by the following definition.

1.1 Definitions. A *spatial geometric context* (more briefly, *spatial theory*) $(\mathrm{Sh}\mathbf{C},\mathscr{U})$ is a topos $\mathrm{Sh}\mathbf{C}$ together with a composable class $\mathscr{U}$ of monomorphisms, called *open immersions*. The poset $\mathscr{U}_{/X}$ of open immersions into $X \in \mathrm{Sh}\mathbf{C}$ are required to satisfy also:

i) stability for base change, i.e. for each $X' \to X$, base change induces a map $\mathscr{U}_{/X} \to \mathscr{U}_{/X'}$;

ii) for each X, $\mathscr{U}_{/X}$ is a complete lattice (i.e. being an open immersion is local);

iii) every covering (epimorphism) of X in $\mathrm{Sh}\mathbf{C}$ can be refined to a covering in $\mathscr{U}_{/X}$.

Let $\mathbf{C}$ be a site for $\mathrm{Sh}\mathbf{C}$. An object $X \in \mathrm{Sh}\mathbf{C}$ is *locally representable* in $\mathbf{C}$ if $\mathscr{U}_{/X}$ is generated by objects of $\mathbf{C}$. The site is *spatial* if it generates $\mathscr{U}$ in the sense that every representable object is locally representable.

Conversely, let $\mathbf{C}$ be a category with a Grothendieck topology generated by a specified class $\mathscr{U}^{\mathrm{aff}}$ of monomorphisms, called 'affine' open immersions, stable for base change, composition, and descent. Then $\mathrm{Sh}\mathbf{C}$ carries a natural structure of a spatial theory for which $\mathbf{C}$ is a spatial site; the open immersions are those monomorphisms that are, after representable base change, exhausted by affine open immersions.

We will always implicitly consider spatial theories equipped with a spatial site, which we may as well assume includes all locally representable objects.

A *spatial geometric morphism* between spatial theories is a geometric morphism of topoi whose pullback preserves open immersions. A flat functor between spatial sites that preserves open immersions and coverings extends to a spatial geometric morphism.

Every object X of a spatial theory has a corresponding small site $\mathrm{Sh}(X) := \mathrm{Sh}(\mathscr{U}_{/X})$, which is by construction *localic*. The locale associated to an object is denoted by the same letter. A morphism of objects induces a geometric morphism of small sites, and hence a continuous map of locales.

If $\mathrm{Sh}\mathbf{C}$ has enough points, then X is actually an honest sober topological space. By Deligne's theorem, this occurs if the Grothendieck topology on a spatial site $\mathbf{C}$ is generated by *finite* coverings. In this series, we will always be working with topoi admitting enough points, and hence confuse locales with spaces.

1.2 Lemma. *Let $f : \mathrm{Sh}\mathbf{C} \to \mathrm{Sh}\mathbf{D}$ be a spatial geometric morphism preserving spatial sites. For each locally representable $X \in \mathrm{Sh}\mathbf{D}$, $f^{-1}X$ is locally representable, and $f^{-1}|\mathscr{U}_{/X}$ is dual to a continuous map $f^{-1}X \to X$ of locales.*

I invite the reader to consult [TV09] for more details of this approach to glueing; though written in a more restricted setting than the above, it is relevant for many examples of interest.

Coverings All coverings used in this paper are really *hypercoverings*, that is, they include the data of the intersections. To be precise, a covering of an object X of a (spatial) site $\mathbf{C}$ is a local isomorphism $X_\bullet \to X$ in PSh$\mathbf{C}$; that is, a morphism that becomes invertible in Sh$\mathbf{C}$. A *member* of a covering is a representable object $X_i \to X_\bullet$ such that $X_i \to X$ is in $\mathscr{U}$.

Algebraic spaces It is also possible to define a broader class of 'algebraic spaces' consisting of any object that can be represented as a colimit of representable objects and open immersions. Many of the things we prove about locally representable objects hold equally well for these algebraic spaces.

2 $\mathbb{F}_1$-schemes and formal schemes

2.1 Review: $\mathbb{F}_1$-algebras

In [Dei08], it was proposed to define $\mathbb{F}_1$-geometry in terms of monoids. In this paper, we use a slight modification: in order to make sense of *vanishing loci*, we want our monoids to have a zero element (as in [CC09]). Thus our $\mathbb{F}_1$-algebras will be commutative algebra objects in the closed symmetric monoidal category of pointed sets.

To put a commutative monoid $(Q,+)$ over $\mathbb{F}_1$, you first write its elements as exponents so that the monoid law can be written multiplicatively:

$$z^X \cdot z^Y = z^{X+Y}.$$

You then adjoin a *sink* or *absorbing element* 0:

$$0 \cdot z^X = 0, \quad \forall X \in Q.$$

The resulting multiplicative monoid can be written $\mathbb{F}_1[z^Q]$ (or simply $\mathbb{F}_1[Q]$ if Q is already written multiplicatively or we are being lazy). Note that even if Q already had a sink, you need to add a new one.

This defines a faithful left adjoint

$$- \otimes \mathbb{F}_1 : \mathbf{Mon} \to \mathrm{Alg}_{\mathbb{F}_1}$$

to the inclusion of the category $\mathrm{Alg}_{\mathbb{F}_1}$ of monoids with zero and homomorphisms that respect zero into the category **Mon** of all monoids. See [CC09, §3.1] (in which the category $\mathrm{Alg}_{\mathbb{F}_1}$ is denoted $\mathfrak{Mon}$) for a more detailed discussion .

2.1 Examples (Fields). An $\mathbb{F}_1$-algebra in which every non-zero element is invertible is called an $\mathbb{F}_1$-*field*. Taking the multiplicative group $K \mapsto K^\times$ of an $\mathbb{F}_1$-field establishes an equivalence between the category of $\mathbb{F}_1$-fields and the category of Abelian groups. In particular, in contrast to the situation over $\mathbb{Z}$, there are non-injective homomorphisms between $\mathbb{F}_1$-fields. Any discrete $\mathbb{F}_1$-field admits a unique homomorphism to $\mathbb{F}_1$, the only 'true' $\mathbb{F}_1$-field.

Any $\mathbb{F}_1$-algebra A has a unique maximal subfield, the *unit* or *coefficient field* $\mathbb{F}_1[A^\times]$, which is just the group of units together with zero. The coefficient field is functorial.

Taking the monoid algebra $-\otimes_{\mathbb{F}_1}\mathbb{Z}$ defines an adjunction

$$\mathrm{Alg}_{\mathbb{F}_1} \leftrightarrows \mathrm{Alg}_{\mathbb{Z}}$$

that allows us to base change $\mathbb{F}_1$-algebras to rings. Both free and forgetful functors commute with localisation.

Finiteness Being a category of commutative monoids in a symmetric tensor category, it is straightforward to make sense of the usual finiteness conditions - most importantly, finite type and finite presentation - for homomorphisms and modules. The Noetherian property also makes sense, and can be defined either by the ascending net condition on ideals or by requiring ideals to be finitely generated. We gather a few basic results here for reference.

2.2 Proposition (Hilbert basis)**.** *Let A be a Noetherian $\mathbb{F}_1$-algebra. Then $A[x]$ is Noetherian.*

Proof. Since $A[x]=\bigvee_{n\in\mathbb{N}}Ax^n$ is a wedge sum of sets, any ideal I is automatically homogeneous. Writing $I=\bigvee_{n\in\mathbb{N}}I_nx^n$, the ascending chain condition implies that $I_n\trianglelefteq A$ stablises for large n. Thus I is finitely generated by $I_0,I_1x,\ldots,I_nx^n$ for large n. □

Note that in the $\mathbb{F}_1$ context, being Noetherian by no means implies that the set of *quotients* satisfies the ascending chain condition - only quotients by *ideals*.

2.3 Corollary. *An $\mathbb{F}_1$-algebra is Noetherian if and only if it is of finite type over its coefficient field.*

Proof. After the Hilbert basis theorem, we only need to show that a Noetherian $\mathbb{F}_1$-algebra is finitely generated over its unit field - this follows because in particular, the maximal ideal $A\setminus A^\times$ is finitely generated. □

In the sequel, we will be led to consider rings that are integral extensions of Noetherian rings, or *integral/Noetherian*.

2.4 Lemma. *Suppose that A is integral/Noetherian. The set of radical ideals of A satisfies the ascending net condition.*

2.2 Review: $\mathbb{F}_1$-schemes

There is a simple way to associate a topological space to an $\mathbb{F}_1$-algebra A (cf. for example [CC09], [Dei05]) - just as in algebraic geometry, you take the set $\mathrm{Spec}\,A$ of prime ideals, topologised with basic open sets given by localisations.

The prime spectrum of A has a unique closed point corresponding to the unique maximal ideal $A\setminus A^\times$ ([Dei05, §1.2]). In other words, affine $\mathbb{F}_1$-schemes are always *local*. This implies that the various other ways one might try to define a covering condition on the category $\mathbf{Sch}^{\mathrm{aff}}_{\mathbb{F}_1}=\mathrm{Alg}^{\mathrm{op}}_{\mathbb{F}_1}$ of affine $\mathbb{F}_1$-schemes are all equivalent - indeed, trivial.

More precisely, for a family $A\to A[f_i^{-1}]$ of localisations, the following are equivalent:

i) $\mathrm{Spec}\,A=\bigcup_i\mathrm{Spec}\,A[f_i^{-1}]$;

ii) f_i is invertible for some i;

iii) $\operatorname{Spec} A \cong \operatorname{Spec} A[f_i^{-1}]$ for some i;

iv) $A \to \prod_i A[f_i^{-1}]$ is a universally effective monomorphism in $\mathrm{Alg}_{\mathbb{F}_1}$;

v) $A \to \prod_i A[f_i^{-1}]$ is effective for descent of modules.

We therefore define the *Zariski topos* $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-schemes to be the presheaf category on $\mathrm{Alg}_{\mathbb{F}_1}^{\mathrm{op}}$. It is a spatial geometric context in the sense of definition 1.1, and the lattice $\mathscr{U}_{/X}$ of open subobjects of an affine object dual to an algebra A is exactly the lattice of open subsets of its prime spectrum $\operatorname{Spec} A$.

The category $\mathbf{Sch}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-schemes is defined in [CC09] as a category of spaces equipped with a sheaf of monoids, locally modelled by affine $\mathbb{F}_1$-schemes. The functor of points

$$\mathbf{Sch}_{\mathbb{F}_1} \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$$

embeds it as the full subcategory of locally representable sheaves in the Zariski topos. This perspective is treated explicitly in [TV09].

One easily makes sense of the usual finiteness conditions (quasi-compact, finite type, locally of finite type, locally Noetherian, etc.). The spectrum of an $\mathbb{F}_1$ algebra A is:

- a point if and only if A is an $\mathbb{F}_1$-field (e.g. 2.1);
- a Noetherian topological space if A is integral/Noetherian (lemma 2.4).

2.5 Aside (Partial addition). It seems likely that to get a true arithmetic over fields of characteristic one, plain monoids are really good enough. Various authors [Dei13, Dur07, Lor12] have introduced categories of objects with a kind of *partial addition* in an attempt to address this.

The deficiency of the plain monoid theory is already visible at the level of geometry, and we will see this crop up a few times throughout this series. A key example is the following: unlike the case of ordinary schemes, our Spec functor does not take finite products to disjoint unions. Indeed, the product $A_1 \times A_2$ of two $\mathbb{F}_1$-algebras A_i has maximal ideal generated by the idempotents $(1,0)$ and $(0,1)$, which corresponds to a point not in either $\operatorname{Spec} A_i$.

The problem here is that we do not have the relation

$$(1,0)+(0,1)=1$$

which in ordinary commutative algebra forces this ideal to equal the whole ring. This problem can be rectified by allowing the addition of $(1,0)$ to $(0,1)$ (but no other additions), which happens automatically if you take the product in the category of monads [Dur07] (and probably, blueprints [Lor12] or sesquiads [Dei13], but I didn't check).

I don't wish to pursue such a generalisation in this paper, though I expect that much of the theory is sufficiently abstract that it runs in a more general setting.

Base change to $\mathbb{Z}$ Base change to $\mathbb{Z}$ commutes with localisation, and there is no descent condition to check, so there is a spatial geometric morphism

$$\mathrm{Sh}\mathbf{Sch}_{\mathbb{Z}} \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1} \qquad \mathbf{Sch}_{\mathbb{F}_1} \to \mathbf{Sch}_{\mathbb{Z}}.$$

The forgetful functor does not preserve coverings, so there cannot be a forgetful functor from $\mathbb{Z}$-schemes to $\mathbb{F}_1$-schemes that preserves the affine objects.

It is possible, following [CC09], to 'glue' the categories of $\mathbb{F}_1$ and $\mathbb{Z}$-schemes together by this morphism, but we won't dwell on that perspective here.

2.3 Pro-discrete

Let A be an $\mathbb{F}_1$-algebra. In the theory of formal schemes, we will want to consider A-modules M equipped with an *A-linear topology*. Such a topology is defined by a filtration of M by A-submodules, which are declared *open*. Indeed, after enlarging such a filtration so that

- the intersection of two open discs is open;
- every disc containing an open disc is open,

these submodules (together with $\emptyset$) are the open sets of a topology on M in the usual sense.

We will *always* assume the following condition:

- the filtration is separated. *Hausdorff*

If, at any point, we find ourselves with a non-separated filtration, we must take the quotient by the intersection of all open submodules, on which the induced topology is Hausdorff.

The feature of linearly topologised modules over $\mathbb{F}_1$-algebras that simplifies the theory considerably, as compared with its counterpart over $\mathbb{Z}$, is the following statement:

2.6 Lemma. *Let M be a Hausdorff, linearly topologised A-module. Then $M \xrightarrow{\sim} \lim M/U$, where U ranges over all open submodules.*

In other words, Hausdorff topological modules are automatically *pro-discrete*, or *complete*, to adapt the terminology of Bourbaki. Ignoring size issues, the argument even allows us to identify the category of Hausdorff linearly topologised A-modules with the category of pro-objects in the category of discrete A-modules whose transition maps are quotients by ideals.[6]

Proof. Let $x \in \lim M/U$, and denote by x_U its image in M/U. If all x_U are 0, then write $\phi(x) = 0$. Otherwise, there exists a U such that $x_U \neq 0$; then the fibre over x_U of

$$M \twoheadrightarrow M/U$$

has a unique element, which we call $\phi(x)$. The unicity implies that it must be a lift of x.

This defines an inverse $\phi : \lim M/U \to M$ to the canonical map. □

The tensor (or smash) product $M_1 \otimes_A M_2$ of linearly topologised modules M_1, M_2 is topologised strongly with respect to the maps

$$e_{x_2} : M_1 \to M_1 \otimes_A M_2, \quad x \mapsto x \otimes x_2 \qquad e_{x_1} : M_2 \to M_1 \otimes_A M_2, \quad x \mapsto x_1 \otimes x$$

for $x_i \in M_i$. In other words, its open submodules are those of the form $M_1 \otimes U_2 \cup U_1 \otimes M_2$, with $U_i \subseteq M_i$ open. The resulting filtration may fail to be Hausdorff, and so the true ('completed') tensor product may be a quotient of the discrete tensor product. It is the limit of the tensor products of the discrete quotients of M_1 and M_2. Since tensor products for us will *always* be completed, we will not introduce a new notation for this construction.

---

[6] Note that this is not possible even for *complete* modules over $\mathbb{Z}$.

A *linearly topologised* or *pro-discrete* $\mathbb{F}_1$*-algebra* is an $\mathbb{F}_1$-algebra A equipped with a (Hausdorff) linear topology as a module over itself. The multiplication $A \otimes_{\mathbb{F}_1} A \to A$ is automatically continuous and open. A pro-discrete $\mathbb{F}_1$-algebra is, up to size issues, the same as a Mittag-Leffler pro-$\mathbb{F}_1$-algebra.

We will also want to make the assumption that

- the product of two open ideals is open. *adicity*

The essential consequence of this condition is that the Rees algebras of A are Banach A-modules, and hence that blow-ups are representable by schemes; see §3.4.

In reality, we usually also assume the existence (at least locally) of a finite type *ideal of definition*

- there exists an open, finitely generated ideal whose powers generate the topology. *admissibility*

which is linguistically reasonable, given that the word 'formal' comes from the formal power series that appear in admissibly topologised rings. This property is not stable under limits.

In fact, apart from the classification by cone complexes, the only place the existence of an ideal of definition is used in this paper is in the proof that separation and propriety of formal schemes can be checked on a reduction (lemma 4.35).

2.7 Definition. A *pro-discrete A-module* is a linearly topologised module over the discrete monoid underlying A such that the action $M \otimes_{\mathbb{F}_1} A \to A$ is continuous. The category of pro-discrete A-modules with continuous A-equivariant homomorphisms is denoted lct_A (the letters standing for 'locally convex topological').

A module M is said to be *adic*, or *Banach*, if its topology is generated by the submodules MU with $U \subseteq A$ an open ideal. Banach modules are stable for tensor product and pullback.

2.8 Examples. The basic examples are those coming from totally ordered Abelian groups $H.$; one topologises the $\mathbb{F}_1$-algebra $\mathbb{F}_1[\![t^{-H}]\!]$ associated to $H^\circ = H_{\leq 0}$ by declaring open the ideals associated to *lower sets* in H°. The sign convention is such that t is topologically nilpotent. Note that, in contrast to the situation over $\mathbb{Z}$, this monoid has the same underlying set as its polynomial counterpart $\mathbb{F}_1[t^{H^\circ}]$.

Such 'formal power series' monoids and arbitary products of discrete monoids satisfy the adicity condition, but typical infinite-dimensional examples like $\mathbb{F}_1[x_1, \overrightarrow{\cdots}]$, an infinite limit of polynomial rings on finite index sets do not.

Base change to $\mathbb{Z}$ In ordinary commutative algebra, the theory of complete linearly topologised rings is *not* equivalent to the theory of pro-discrete rings. When discussing formal schemes over $\mathbb{Z}$, we will use the latter as our definition; that is, a pro-discrete ring is a pro-object of $\mathrm{Alg}_{\mathbb{Z}}$ with surjective transition maps.

The base change functor on discrete algebras has a natural pro-extension

$$\mathrm{Pro}^{\mathrm{ML}}\mathrm{Alg}_{\mathbb{F}_1} \to \mathrm{Pro}^{\mathrm{ML}}\mathrm{Alg}_{\mathbb{Z}}$$

that commutes with tensor products and manifestly preserves 'admissibility'. However, since the forgetful functor $\mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Alg}_{\mathbb{F}_1}$ does not preserve quotients by ideals, this functor does not have a right adjoint, i.e. there is no forgetful functor from pro-discrete rings to pro-discrete $\mathbb{F}_1$-algebras.

2.4 Some definitions of formal $\mathbb{F}_1$-schemes

One extends the definition of spectrum to linearly topologised $\mathbb{F}_1$-algebras A by analogy with the traditional setting [Gro60, **I**.10], that is, by taking the set of *open* prime ideals. The principal open sets correspond to *completed* localisations of A, which are defined exactly as over $\mathbb{Z}$:

$$\lim_J A/J = A \to A\{f^{-1}\} := \lim_J A/J[f^{-1}].$$

The formal prime spectrum is a 'pro-discretely monoidal space', which I will break with tradition by denoting $\operatorname{Spec} A$. This will not cause ambiguity, as we will never take the spectrum of *all* prime ideals of a non-discrete ring.

Since the maximal ideal is always open, the formal spectrum of a linearly topologised ring continues to be local. In other words, there are no non-trivial coverings of affine formal schemes over $\mathbb{F}_1$.

The equivalent definitions of the (trivial) covering condition on the opposite $\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1}$ to the category of admissible $\mathbb{F}_1$-algebras still hold good in the pro-discrete regime - indeed, we can even add

vi) $A \to \prod_i A[f_i^{-1}]$ is effective for descent of pro-discrete modules.

vii) $A \to \prod_i A[f_i^{-1}]$ is effective for descent of Banach modules.

Thus any of these - now seven - conditions define the same spatial theory $\mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}$, which is the presheaf category on $\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1}$ with open immersions dual to completed localisations. It carries a tautological sheaf $\mathscr{O}$ of adic $\mathbb{F}_1$-algebras.

We may therefore give two equivalent definitions of formal schemes:

2.9 Definitions (Formal schemes)**.** A formal $\mathbb{F}_1$-scheme is

i) a space with a sheaf $\mathscr{O}$ of adic $\mathbb{F}_1$-algebras, locally isomorphic to the spectrum of an adic (admissible) $\mathbb{F}_1$-algebra.

ii) a locally representable presheaf on $\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1}$.

The category of formal $\mathbb{F}_1$-schemes is denoted $\mathbf{FSch}_{\mathbb{F}_1}$.

2.10 Aside. There are actually some subtleties involved in making sense of a 'sheaf of adic $\mathbb{F}_1$-algebras'. The theory of pro-discrete $\mathbb{F}_1$-algebras is not a finite products theory, and so the category of pro-discrete $\mathbb{F}_1$-algebras internal to the sheaf topos of a space X is *not* the same as the category of colimit-preserving functors from $\mathrm{Sh}(X)$ into the category of pro-discrete $\mathbb{F}_1$-algebras. This fact will be familiar to students of the ℓ-adic cohomology.

Spaces defined in terms of affine pieces like formal schemes do not suffer from this ambiguity, so either approach would be sufficient for the purposes of this definition. However, note that even for formal schemes, since admissibility is not stable for limits, general section spaces of $\mathscr{O}$ need not admit ideals of definition.

The inclusion of the subcategory of discrete algebras gives a spatial geometric morphism

$$\mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1} \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1} \qquad \mathbf{Sch}_{\mathbb{F}_1} \to \mathbf{FSch}_{\mathbb{F}_1}.$$

Via the fully faithful composite of the Yoneda embedding and the pushforward functor

$$\mathbf{FSch}_{\mathbb{F}_1} \hookrightarrow \mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1} \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1},$$

it is also possible to consider formal schemes as certain locally *ind-representable* objects of $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$. Indeed, by definition the opposite category to that of admissibly topologised $\mathbb{F}_1$-algebras is a full subcategory of $\mathrm{Ind}\mathbf{Sch}^{\mathrm{aff}}_{\mathbb{F}_1}$ whose objects, among other conditions, have transition maps finitely presented nilpotent embeddings. For simplicity we also take this perspective on formal schemes over $\mathbb{Z}$.

The definition of blow-ups, and hence of rigid analytic spaces, is not local for $\mathrm{Sh}\mathbf{Sch}$, so we will still need to use the larger topos $\mathrm{Sh}\mathbf{FSch}$ in the sequel.

The categories lct_- extend to a stack $\mathrm{QC} = \mathrm{QC}_{\mathbf{FSch}}$ on the formal topos; the categories Ban_- define a full substack (def. 2.7). If f is a qcqs morphism of formal schemes, the corresponding pullback morphism f^* on QC has a right adjoint f_*. A qcqs morphism is representable by schemes in $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$ - a property the literature usually calls *adicity* - if and only if this pushforward functor preserves the substack of Banach modules.

Base change to $\mathbb{Z}$ The category of affine formal schemes over $\mathbb{Z}$ is, by our definition, opposite to the full subcategory of Mittag-Leffler pro-rings

$$\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{Z}} \hookrightarrow \mathrm{ProAlg}^{\mathrm{op}}_{\mathbb{Z}}$$

defined by the criteria of adicity and admissibility. Note that this is slightly different from the traditional theory expounded in terms of topological algebra in [Gro60, **1**.1.10], though under the axiom of dependent choice the two approaches can be shown to be equivalent at least for first countably topologised algebras.

The base change functor $\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1} \to \mathbf{FSch}^{\mathrm{aff}}_{\mathbb{Z}}$ is the restriction of the ind-extension of the base change for schemes, and it gives rise, as in the former case, to a spatial geometric morphism

$$\mathrm{Sh}\mathbf{FSch}_{\mathbb{Z}} \to \mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1} \qquad \mathbf{FSch}_{\mathbb{F}_1} \to \mathbf{FSch}_{\mathbb{Z}}.$$

2.5 Formal completion

The initial remarks of this section are valid for $\mathbb{F}_1$ and for $\mathbb{Z}$. Note that in the latter case our definitions deviate a little (see immediately above) from the traditional ones, under which some of the assertions of this section are false.

Let ${}^Z\mathbf{FSch}$ denote the category whose objects are pairs (X,Z) consisting of a formal scheme over $\mathbb{F}_1$ or $\mathbb{Z}$ together with a closed, finitely presented algebraic subscheme Z, and whose morphisms $f : X_1 \to X_2$ satisfy $Z_1 \subseteq (f^{-1}Z_2)^{\mathrm{red}}$. The isomorphism class of an object (X,Z) of ${}^Z\mathbf{FSch}$ depends only on X and the underlying set of Z.

The forgetful functor

$${}^Z\mathbf{FSch} \to \mathbf{FSch}$$

has adjoints on both sides. The right adjoint, which marks an entire formal scheme, has a further right adjoint: *formal completion.*

The *formal completion of X along Z* is, as an ind-scheme, the inductive limit

$$\hat{X}_Z := \operatorname*{colim}_{Z' \subseteq Z} Z'$$

of closed subschemes with set-theoretic support in Z. Formal completion is idempotent. If X is a scheme, then the formal completion along Z may be computed as the double complement of Z in the Heyting algebra of subobjects of X in the Zariski topos.

To see that $\hat{X}_Z$ is a formal scheme according to our definitions (§2.3,2.4), we will need to show that its co-ordinate algebra is adic. We declare open any quasi-coherent ideal sheaf $I \trianglelefteq \mathcal{O}_X$ cosupported on Z, and write $\hat{\mathcal{O}}_{X,Z}$ for the Hausdorff quotient of the linear topology on $\mathcal{O}_X$ thus obtained. It is a pro-discrete quasi-coherent sheaf on X, the limit of all quotients of $\mathcal{O}_X$ supported on Z.

Since cosupports remain unchanged under taking powers of an ideal, $\hat{\mathcal{O}}_{X,Z}$ is adic, and hence its spectrum $\hat{X}_Z$ is a (marked) formal scheme. Its ideal of definition given by the ideal defining Z.

2.11 Proposition (Formal completion commutes with base change)**.** *Let (Y,Z) be a marked formal scheme, $f:X\to Y$ any morphism (including along* $\operatorname{Spec}\mathbb{Z}\to\operatorname{Spec}\mathbb{F}_1$*). The square*

$$\begin{tikzcd}
\hat{X}_{f^*Z} \arrow[r] \arrow[d] & X \arrow[d] \\
\hat{Y}_Z \arrow[r] & Y
\end{tikzcd}$$

is Cartesian.

Proof. Follows from the existence of an ideal of definition. □

2.12 Example. In the extreme example where $Z=X$, the formal completion is X itself. The 'next most faithful' formal completion of X comes from setting Z the *boundary* of X, that is, the union of all divisors. If X is a locally Noetherian (resp. Noetherian) $\mathbb{F}_1$-scheme, the result is a locally Noetherian (resp. Noetherian) formal $\mathbb{F}_1$-scheme. This is in marked contrast to the same operation in algebraic geometry, which generally yields a huge mess.

Functorial algebraisation If A is a topological $\mathbb{F}_1$-algebra, let us write $A^?$ for the 'forgetful' underlying discrete $\mathbb{F}_1$-algebra. It defines a functor

$$?:\mathbf{FSch}^{\mathrm{aff}}\to {}^Z\mathbf{Sch}^{\mathrm{aff}}$$

which marks all open ideals. It is left adjoint, and right inverse, to formal completion.

Of course, for ordinary schemes one hits a wall as soon as one tries to globalise this functor. But, stupidly, since over monoids there are no non-trivial coverings of formal schemes, it actually extends to a colimit-preserving (but not left exact) functor

$$?:\mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}\to \mathrm{Sh}^Z\mathbf{Sch}_{\mathbb{F}_1} := \mathrm{PSh}^Z\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff}}$$

left adjoint and right inverse to formal completion.

2.13 Definition. A formal scheme X is said to be *algebraisable* if there exists a marked scheme (Y,Z) and an isomorphism of (unmarked) formal schemes $\hat{Y}_Z\cong X$; these data are called an *algebraisation* of X.

The forgetful sheaf ${}^{?}X$ is a natural candidate for an algebraisation of X: if ${}^{?}X$ is a scheme, then it is a *functorial* algebraisation of X. We are led naturally to the following strengthening of the algebraisability question:

2.14 Question. *When does* ? *take formal schemes to schemes?*

We give a complete answer to this question for integral formal schemes in §7.6.

2.6 Markings

In the sequel, it will be useful to have a category parametrising *marked* formal schemes, that is, formal scheme marked along a family of closed formal subschemes. This is a variation on the definition of the category ${}^{Z}\mathbf{FSch}$ (§2.5).

2.15 Definition. A *marked formal scheme* is a pair (X,Z) consisting of a formal scheme X and a family of finitely presented closed formal subschemes Z. A morphism of pairs is a morphism $f: X_1 \to X_2$ of formal schemes such that $(f^{-1}Z_2)^{\mathrm{red}} \subseteq Z_1$.

The category of marked schemes, resp. formal schemes is denoted ${}_{Z}\mathbf{Sch} \hookrightarrow {}_{Z}\mathbf{FSch}$.

The isomorphism class of (X,Z) depends only on the underlying reduced formal schemes Z^{red} of Z. A marked formal scheme (X,Z) has a canonical 'maximal' representative in its isomorphism class, in which Z is replaced with the family of all finitely presented closed subschemes with set-theoretic support in a finite union of members of Z.

If, in fact, $f^{-1}Z_2 = Z_1$, we say that f is *represented by the formal scheme* X_1 over (X_2,Z_2); the slice category of represented morphisms is simply $\mathbf{FSch}_{X_2}$.

Let us call a family $(X_\bullet, Z_\bullet) \to (X,Z)$ a *covering* if it is represented by formal schemes and $X_\bullet \to X$ is a Zariski covering. If the formal schemes involved are affine, it is equivalent to ask simply that one of the maps is an isomorphism of pairs. This system of coverings defines a topos $\mathrm{Sh}_{Z}\mathbf{FSch}$, the *marked formal topos*. It is a spatial theory (def. 1.1) with represented Zariski-open immersions, and the locally representable objects are precisely the marked formal schemes.

Adjoints The forgetful functor

$$ {}_{Z}\mathbf{FSch} \to \mathbf{FSch} $$

has a fully faithful right adjoint, which associates to the formal scheme X the pair $(X,\emptyset)$, and left adjoint, which associates (X,X). The right adjoint has a further right adjoint $(X,Z) \mapsto X \setminus Z$, which commutes with coverings. These four adjoints, from left to right:

$$\begin{aligned} (X,X) &\mapsfrom X \\ (X,Z) &\mapsto X \\ (X,\emptyset) &\mapsfrom X \\ (X,Z) &\mapsto X \setminus Z \end{aligned}$$

induce three essential geometric morphisms

$$\begin{tikzcd} \mathrm{Sh}\mathbf{FSch} \arrow[r, shift left=1.5] & \mathrm{Sh}_{Z}\mathbf{FSch}; \arrow[l] \arrow[l, phantom, ""] \end{tikzcd}$$

the one going from right to left is surjective, and the other two are sections.

More generally, if $(Y,Z) \in {}_Z\mathbf{FSch}$ is any object, the right adjoint to

$${}_Z\mathbf{FSch}_{(Y,Z)} \to \mathbf{FSch}_Y$$

takes X to $(X, Z \times_Y X)$. The other adjoints have the same formulas.

3 Elementary properties of morphisms

Here I review some easy properties of morphisms of formal schemes, in generalisation of the concepts presented in [Gro60, **II**] for schemes over $\mathbb{Z}$. In particular, we make a special study of projective morphisms §3.3 and blow-ups §3.4, a good understanding of which is critical to the definitions of both overconvergence and rigid analytic geometry [Mac15].

Unless otherwise noted, all definitions are valid over $\mathbb{F}_1$ and $\mathbb{Z}$ (and are mostly standard in the latter case); I therefore suppress the subscripts denoting the base. Unless otherwise noted, 'stable for base change' includes the base change $\mathbb{F}_1 \to \mathbb{Z}$.

3.1 Embeddings and immersions

The passage from $\mathbb{Z}$ to $\mathbb{F}_1$ presents only two complications, both already visible in the class of *embeddings*.

The first issue is that surjective morphisms of monoids need not be determined by their kernel. Dually, we find that embeddings - inclusions of subschemes cut out by equations - need not be *closed*. This is a manifestation of the same issue that necessitates an alternative approach to separation and propriety in §4.

The second issue is of a more pathological nature (i.e. it should probably fixed by any 'true' theory of $\mathbb{F}_1$, should such a thing exist): affine monoid schemes are necessarily connected, so the inclusion of a disconnected closed subscheme of a connected scheme cannot be affine. This makes it somewhat difficult to guarantee the existence of embedded closures for arbitrary immersions.

For the purposes of this paper, luckily, we are able to completely sidestep having to address the second issue; I only formulate a couple of unanswered questions 3.4, 3.22.

3.1 Definitions (Embeddings)**.** An affine morphism $X \to Y$ is said to be a *formal embedding* if for each closed algebraic subscheme $X_0 \hookrightarrow X$, $\mathscr{O}_Y \to \mathscr{O}_Y(X_0)$ is surjective. Over $\mathbb{F}_1$, this condition is the same as asking that $\mathscr{O}_Y \to \mathscr{O}_Y(X)$ be surjective. It is moreover an *embedding* if it is representable by schemes. Affine formal embeddings are monic.

A quasi-compact monomorphism $X \to Y$ is a (formal) embedding if it can be covered by affine (formal) embeddings; that is, if locally on Y there are coverings

$$U_\bullet \to X_\bullet \twoheadrightarrow X$$

in Sh**FSch** with $U_\bullet \to X_\bullet$ an open immersion, $U_\bullet \twoheadrightarrow X$ an open cover and each $X_i \to Y$ an affine (formal) embedding.

The partially ordered set of embedded (resp. formally embedded) subschemes of Y is denoted $\mathfrak{Z}(Y)$ (resp. $\hat{\mathfrak{Z}}(Y)$). Since any formal embedding is ind-representable by embeddings, $\hat{\mathfrak{Z}}$ is generated under filtered suprema by $\mathfrak{Z}$.

Over $\mathbb{Z}$, every embedding is a closed immersion associated to some ideal, and in particular, affine. One can therefore define unions of embeddings by intersecting ideals, and $\mathfrak{Z}$ has finite joins. The same follows for $\hat{\mathfrak{Z}}$, since it is generated by $\mathfrak{Z}$.

Over $\mathbb{F}_1$, embeddings can fail to be closed, and finite joins in $\mathfrak{Z}$, even when they exist, can fail to be set-theoretic unions. They can also fail to be affine; see example 3.5.

3.2 Definition (Embedded image)**.** A *formally embedded image* for a morphism $X \to Y$ is an embedded subscheme $\mathrm{cl}(X/Y)$ of Y initial among those through which X factors.

An open immersion $X \hookrightarrow Y$ is said to be *dense*, or more precisely *scheme-theoretically dense*, if Y is a formally embedded closure of X in Y. A dense open immersion is always dense in the sense of point-set topology, but not vice versa.

One can construct a formally embedded image of an affine morphism $f : X \to Y$ via the formula

$$\mathrm{cl}(X/Y) := \mathrm{Spec}\left(\lim_{X_0 \subseteq X} \mathrm{Im}[\mathscr{O}_Y \to \mathscr{O}_Y(X_0)]\right) \tag{3}$$

where the limit is over closed algebraic subschemes of X. Over $\mathbb{F}_1$, this limit is the same as the image of $\mathscr{O}_Y$ in $\mathscr{O}_Y(X)$ with the *subspace* topology.

3.3 Lemma. *Let $X \to Y$ be an affine morphism. Then $\mathrm{cl}(X/Y)$ as defined by formula (3) is a formally embedded image of X/Y.*

Proof. Both the definition of embedding and the stated formula (3) are local on the target, so we may assume Y is affine.

Let $Z \hookrightarrow Y$ be an embedded subscheme through which X factors. If we are over $\mathbb{Z}$, Z is affine. Otherwise, let $\bigcup_i Z_i$ be an affine covering of Z. Since affine schemes over $\mathbb{F}_1$ have no non-trivial coverings, we must have $Z_i \times_Y X = X$ for a single index i.

Either way, the embedded image of X in Y is affine, and hence computed by (3). □

If X and Y are both schemes, the formally embedded image is in fact an embedded subscheme. However, for formal schemes $\mathrm{cl}(X/Y) \to Y$ is rarely representable by schemes, even when $X \hookrightarrow Y$ is an open immersion.

We would like be able to construct formally embedded images for more general quasi-compact morphisms $X \to Y$, that is, *left adjoints*

$$\mathrm{cl}(-/Y) : \mathbf{FSch}_Y \to \hat{\mathfrak{Z}}(Y)$$

to the inclusion. By lemma 3.3, this adjoint exists on the subcategory $\mathbf{FSch}_Y^{\mathrm{aff}}$ and for any formal scheme over $\mathbb{Z}$.

For more general formal schemes over Y, one can at least define a pro-adjoint

$$\mathbf{FSch}_Y \to \mathrm{Pro}\hat{\mathfrak{Z}}(Y);$$

that is, the formally embedded closure always makes sense as a pro-object of $\mathbf{FSch}_Y$. The pro-adjoint would be the extension of an ordinary adjoint if one could take arbitrary limits of embeddings. However, since embeddings can fail to be affine it is not clear that this is possible.

The matter would be settled by a positive answer to the following question, inspired by the example 3.5:

3.4 Question. *Is every embedding affine in the category of schemes relative to monads (or blueprints, or sesquiads...)?*

3.5 Example ($\hat{\mathfrak{z}}(\mathbb{A}^1)$). The formally embedded subobject poset of the affine line $\mathbb{A}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$ has the form, increasing from left to right:

$$\begin{tikzcd}
 & \{1\} \arrow[r, dash] & \{1\} \sqcup \{0\} \arrow[r, dash] & \{1\} \sqcup \{0\}^2 \arrow[r, dash, dotted] & \{1\} \sqcup \hat{\mathbb{A}}^1_0 \arrow[r, dash] & \mathbb{A}^1 \\
\emptyset \arrow[r, dash] \arrow[ur, dash] & \{0\} \arrow[r, dash] \arrow[ur, dash] & \{0\}^2 \arrow[r, dash, dotted] \arrow[ur, dash] & \hat{\mathbb{A}}^1_0 \arrow[ur, dash] & &
\end{tikzcd}$$

The disjoint union of the origin and the non-closed embedded point 1 in is embedded, but being disconnected, is not affine. Note that its inclusion is a bijection, but not a topological immersion. It is an embedded open subset of its affinisation $\mathrm{Spec}(\mathbb{F}_1 \times \mathbb{F}_1)$, which itself is no longer immersed.

This counterexample vanishes after enlarging the category of monoids to include certain monads with non-trivial addition as in e.g. 2.5. After base change to $\mathbb{Z}$, the embedding of course becomes closed.

3.6 Definition (Immersions)**.** A morphism is a *(formal) immersion* if it can be written as an open immersion followed by a (formal) embedding.

An immersion of $\mathbb{F}_1$-schemes need not be a topological immersion; see example 3.5.

In the case that $X \hookrightarrow Y$ is an immersion, we also call $\mathrm{cl}(X/Y)$ the *formally embedded closure* of X in Y. A (formal) immersion $X \hookrightarrow Y$ is a (formal) embedding if and only if for all affine (formal) immersions $U \hookrightarrow Y$ factoring through X, the formally embedded closure $\mathrm{cl}(U/Y)$ is also contained in X.

3.7 Proposition. *An immersion is open in the embedded image of its affinisation. In particular, an affine immersion is open in its embedded closure.*

Proof. Same proof as [Sta14, 01P9]. □

3.8 Proposition (Stability)**.** *The class of (formal) immersions is stable for base change and descent. The class of (formal) embeddings is stable for composition, base change, and descent.*

3.2 Integral morphisms and relative normalisation

Let A be an $\mathbb{F}_1$-algebra, and let B be a *finite* A-algebra, that is, an algebra that is finitely generated as an A-module. It follows that for every $f \in B$, either $f^n \in A$ for some $n > 0$, or the set $\{f, f^2, \ldots\}$ is finite. In other words, f satisfies a *monic equation*

$$f^n = c_i f^i, \quad i < n, c_i \in A$$

over A. One can therefore define *integral closure* of pairs of $\mathbb{F}_1$-algebras in much the same way as for commutative rings.

3.9 Definition. An affine morphism $X \to Y$ in Sh**FSch** is said to be *finite* if $\mathcal{O}_Y(X)$ is a finite Banach $\mathcal{O}_Y$-algebra; that is, locally on Y there is a topological quotient $p : \mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{O}_Y(X)$ for some $n \in \mathbb{N}$. We say that a morphism is *integral* if it is a limit of finite morphisms in the category of formal schemes adic over Y.

An affine morphism is said to be *formally finite* if locally on Y there is a module homomorphism $p : \mathcal{O}_Y^{\oplus n} \to \mathcal{O}_Y(X)$ that surjects onto every discrete quotient of $\mathcal{O}_Y(X)$. Over $\mathbb{F}_1$, this is equivalent to surjectivity of p itself. A morphism is *formally integral* if it is integral over a formally finite morphism.

A *relative normalisation* of an affine morphism $X \xrightarrow{f} Y$ is an integral morphism $\nu_f Y \to Y$ initial among those factoring f. A relative normalisation can be constructed by taking the integral closure of the image of $\mathcal{O}_Y$ inside $f_*\mathcal{O}_X$.

3.10 Aside. Since embeddings can fail to be affine, any reasonable definition would also allow finite morphisms to be non-affine. Following the template of definition 3.1, we might say that a general quasi-compact morphism is finite if it can be covered by affine finite morphisms.

For the purposes of this paper, we can survive without treating non-affine finite morphisms.

A finite morphism $X \to Z$ followed by a formal embedding $Z \hookrightarrow Y$ is formally finite: the surjection $\mathcal{O}_Y(Z)^{\oplus n} \twoheadrightarrow \mathcal{O}_Y(X)$ induces a module homomorphism $\mathcal{O}_Y^{\oplus n} \to \mathcal{O}_Y(X)$ satisfying the requirement of the definition. Of course, by definition an integral morphism followed by a formal embedding is formally integral.

Conversely, by taking the formally embedded image in Y, any formally finite (resp. integral) affine morphism $X \to Y$ can be written as a finite (resp. integral) morphism followed by an affine formal embedding. By the same logic, a formal embedding followed by a finite (resp. integral) morphism can be rewritten in this way, and is hence, in particular, formally finite (resp. integral).

3.11 Lemma. *Any formally finite affine morphism can be written as a finite morphism followed by an affine formal embedding. Any composite of finite morphisms and affine formal embeddings is formally finite.*

Any formally integral affine morphism can be written as an integral morphism followed by aa affine formal embedding. Any composite of integral morphisms and affine formal embeddings is formally integral.

3.12 Proposition (Stability)**.** *The classes of finite, integral, formally finite, and formally integral morphisms are stable for composition, base change, and descent. If gf is finite, resp. integral, resp. formally finite, resp. formally integral, then so is f.*

Divisorial markings For rigid analytic geometry, we will need a slightly refined version of the relative normalisation, adapted to the full subcategory ${}_Z\mathbf{FSch}^{\mathrm{div}} \subset {}_Z\mathbf{FSch}$ of *divisorially marked* formal schemes, that is, marked formal schemes for which Z is a collection of locally principal subschemes. The isomorphism class of an object (Y, Z) of ${}_Z\mathbf{FSch}^{\mathrm{div}}$ is determined by the formal scheme Y together with the multiplicative subset $S_Z \subseteq \mathcal{O}_Y$ comprised of sections whose vanishing locus has radical contained in Z.

We may therefore write this object of ${}_Z\mathbf{FSch}^{\mathrm{div}}$ as $(Y; \mathcal{O}_Y[S_Z^{-1}])$, where $\mathcal{O}_Y[S_Z^{-1}]$ denotes the localisation in the category of all (not necessarily pro-discrete) topological modules.[7]

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[7] Strictly speaking, over $\mathbb{Z}$ one must be a little careful in making sense of this localisation. It is good enough to consider it as a certain ind-object in the category of Banach $\mathcal{O}_Y$-modules.

3.13 Definitions. An integral morphism $X \to Y$ is an *isomorphism away from* Z, or Z-*admissible*, if $\mathcal{O}_Y[S_Z^{-1}] \to \mathcal{O}_X[S_Z^{-1}]$ is an isomorphism.

We say that the pair $(Y, \mathcal{O}_Y[S_Z^{-1}])$ is *relatively normal*, or that Y is *normal along* Z, if Z are Cartier divisors and $\mathcal{O}_Y$ is integrally closed in $\mathcal{O}_Y[S_Z^{-1}]$. Equivalently, (Y,Z) is relatively normal if and only if Z is Cartier and any Z-admissible finite morphism into Y is an isomorphism.

Let us denote the full subcategories of ${}_Z\mathbf{FSch}^{\mathrm{div}}$ whose objects are marked along Cartier divisors, resp. are relatively normal, by ${}_Z\mathbf{FSch}^{\mathrm{inv}}$, resp. ${}_Z\mathbf{FSch}^{\nu}$.

Given a pair $(Y, \mathcal{O}_Y[S_Z^{-1}]) \in {}_Z\mathbf{FSch}^{\mathrm{div}}$, we can construct a *relative normalisation*, or *normalisation of* Y *along* Z, in two stages:

i) replace $\mathcal{O}_Y$ with its image in $\mathcal{O}_Y[S_Z^{-1}]$; *invert* Z

ii) pass to the integral closure inside $\mathcal{O}_Y[S_Z^{-1}]$. *separate crossings along* Z

The first stage yields an embedded subscheme of Y, and the second yields an integral morphism $\nu_Z Y \to Y$ whose embedded image is that produced by the first. These two stages constitute right adjoints to the inclusions

$$ {}_Z\mathbf{FSch}^{\nu} \leftrightarrows {}_Z\mathbf{FSch}^{\mathrm{inv}} \leftrightarrows {}_Z\mathbf{FSch}^{\mathrm{div}} $$

It satisfies the following universal property: any finite morphism $X \to Y$ that is an isomorphism away from Z factorises uniquely

$$ \nu_Z Y \to X \to Y $$

the relative normalisation.

3.3 Projective morphisms

One can define, as for usual schemes over $\mathbb{Z}$, the Proj of a positively graded $\mathbb{F}_1$-algebra A - that is, an algebra object in the category of $\mathbb{N}$-indexed families of pointed sets - in terms of homogeneous prime ideals. Even without this description, one can easily define the principal affine subsets of $\operatorname{Proj} A$, in the usual way, as the spectra of the degree zero piece $A[f^{-1}]_0$ of the $\mathbb{Z}$-graded localisation.

We lift the usual definitions of quasi-projective, projective, ample and very ample invertible sheaf from [Gro60, **II**].

In particular, for any quasi-coherent Banach module V, we can define as usual a projective bundle

$$ \mathbb{P}(V) := \operatorname{Proj}(\operatorname{Sym}^\bullet V), $$

qcqs when V is of finite type, and this 'generates' the definition of projectivity. By choosing a finitely presented covering $V^{\mathrm{fp}} \twoheadrightarrow V$, one can always write a projective bundle as a finitely presented projective bundle $\mathbb{P}(V^{\mathrm{fp}})$ composed with an affine embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V^{\mathrm{fp}})$.

3.14 Lemma. *Let* V *be a quasi-coherent Banach module on* Y, $f : X \to Y$ *a morphism. Then* $\mathbb{P}(f^*V) \cong \mathbb{P}(V) \times_Y X$.

3.15 Lemma. *The diagonal of a projective bundle is an affine embedding.*

Proof. By direct calculation, which applies equally over $\mathbb{Z}$ and over $\mathbb{F}_1$. □

3.16 Definition. A morphism is said to be (formally) *projective*, resp. *integral/projective* if it is (formally) finite, resp. integral, over a projective bundle. We may always assume that the projective bundle is finitely presented.

A projective morphism $\mathrm{Proj}\, A \to Y$ to a divisorially marked formal scheme $Y \in {}_Z\mathbf{FSch}^{\mathrm{div}}$ is an *isomorphism away from Z*, or *Z-admissible*, if $A_k[S_Z^{-1}]$ is an invertible $\mathcal{O}_Y[S_Z^{-1}]$-module for $k \gg 0$. The definition extends to integral/projective morphisms via def. 3.13.

3.17 Proposition (Stability)**.** *Projective and integral/projective morphisms, as well as their formal variants, are stable for composition and base change.*

The diagonal of a formally projective morphism is an affine embedding.

Proof. To show stability under composition for projective morphisms, we must show that a finitely presented projective bundle $\mathbb{P}_X(V) \to X$ followed by a finite morphism $X \to Y$ may be written the other way around, and hence is projective. Let V_Y be a model for V over Y. Then the square

$$\begin{tikzcd}
\mathbb{P}_X(V) \arrow[r] \arrow[d] & \mathbb{P}_Y(V_Y) \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

is Cartesian, whence the result. The proof for the more general classes proceeds in the same manner, *mutatis mutandi*. □

By definition, a morphism is projective if and only if it is the Proj of a finitely generated graded $\mathcal{O}_Y$-algebra whose piece in degree zero is finite.

If, more generally, A is a graded quasi-coherent $\mathcal{O}_Y$-algebra finitely generated over its degree zero piece A_0, and A_0 is integral (resp. formally finite, resp. formally integral) over $\mathcal{O}_Y$, then $\mathrm{Proj}\, A \to Y$ is integral/projective (resp. formally projective, resp. formally integral/projective).

To attempt to prove a converse, one might consider the affinisation

$$X \to \mathrm{Spec}\, f_*\mathcal{O}_X \to Y$$

of an integral/projective morphism $f : X \to Y$. By proposition 3.17, $X \to \mathrm{Spec}\, f_*\mathcal{O}_X$ is integral/projective. We would like to know that $f_*\mathcal{O}_X$ is an integral $\mathcal{O}_Y$-algebra. Since f_* commutes with filtered colimits, it would be enough to know that when f is *projective*, $f_*\mathcal{O}_X$ is *finite*.

This brings us to the question of finiteness of global sections over projective morphisms:

3.18 Question. *Let $f : X \to Y$ be projective. Is $f_*\mathcal{O}_X$ a finite $\mathcal{O}_Y$-module?*

A positive answer to this question in general would imply:

$$\begin{array}{rcl} \text{projective \& affine} & \Rightarrow & \text{finite} \\ \text{integral/projective \& affine} & \Rightarrow & \text{integral} \end{array}$$

Indeed, the second statement would follow from the first and the fact that any affine morphism is a limit of finite type affine morphisms. Of course, there cannot be any analogue of these implications for formally projective morphisms.

An attack on this question for $\mathbb{F}_1$ would take us far beyond the scope of this paper. For $\mathbb{Z}$, if f is pseudo-coherent (for example, if Y is locally Noetherian), it is a consequence of the much more general projective pushforward theorem [BGI71, III.2.2]. In the special case of pushing forward the structure sheaf, we can easily make do with a little less technology:

3.19 Lemma. *Let $f : X \to Y$ be a projective morphism of formal schemes over $\mathbb{Z}$. Then $f_*\mathscr{O}_X$ is an integral $\mathscr{O}_Y$-algebra.*[8]

Proof. The question being local on Y, suppose $Y = \operatorname{Spec}\mathscr{O}(Y)$ is affine. Then X is a closed subscheme of a projective space $\mathbb{P}^r_Y$. Let $\{A_i\}_{i\in I}$ be the filtered set of pairs consisting of a Noetherian subring A of $\mathscr{O}(Y)$ and a closed subscheme $X_i \subseteq \mathbb{P}^r_i$ such that $X \subseteq X_i \times_i \mathbb{P}^r_Y$. Then $X \to X_i$ is affine, and $X = \lim_i X_i$.

Pick $Y_0 = \operatorname{Spec} A_0$. For every $i \to 0$, we have commuting squares

$$\begin{tikzcd}
X_i \arrow[r, "g_i"] \arrow[d, "f"'] & X_0 \arrow[d, "f"] \\
Y_i \arrow[r, "g_i"] & Y_0
\end{tikzcd}$$

We will calculate global sections on Y_0.

Since the transition maps are affine,

$$\operatorname*{colim}_i g_{i*}\mathscr{O}_{X_i} \xrightarrow{\sim} g_*\mathscr{O}_X$$

in the category of quasi-coherent sheaves on $\mathbb{P}^r_0$. Since f_* commutes with filtered colimits, we may push forward this isomorphism to Y_0 and calculate

$$\operatorname*{colim}_i g_{i*}\mathscr{O}_{Y_i} \cong g_*\mathscr{O}_Y \longrightarrow f_*g_*\mathscr{O}_X \cong \operatorname*{colim}_i f_*g_{i*}\mathscr{O}_{X_i}$$

as a filtered colimit of homomorphisms $g_*\mathscr{O}_{Y_i} \to g_*f_{i*}\mathscr{O}_{X_i}$.

In order for this homomorphism to be integral, it will be enough for each term $\mathscr{O}_{Y_i} \to f_*\mathscr{O}_{X_i}$ to be finite. But this follows from the finiteness of cohomology over projective morphisms in the Noetherian case. ☐

3.20 Corollary. *An affine and integral/projective morphism between formal schemes over $\mathbb{Z}$ is integral.*

3.21 Example. Any affine embedding $Z \hookrightarrow Y$ is projective - in fact, a projective bundle $\mathbb{P}(\iota_*\mathscr{O}_Z)$. However, this fails for our standard example of a non-affine embedding - a finite union $Z = \sqcup_{i=1}^k Z_k$ of disjoint affine embeddings $Z_i \hookrightarrow Y$ - which is actually only an embedded open subset of

$$\mathbb{P}(\iota_*\mathscr{O}_Z) = \mathbb{P}\left(\prod_{i=1}^k \iota_*\mathscr{O}_{Z_i}\right) = \operatorname{Spec}\left(\prod_{i=1}^k \iota_*\mathscr{O}_{Z_i}\right).$$

On the other hand, in this case Z is the projectivisation of the finite Banach module $\bigoplus_{i=1}^k \mathscr{O}_{Z_i}$. In particular, this gives an example where a projective morphism may fail to be isomorphic to the Proj of its homogeneous co-ordinate ring.

3.22 Question. *Is every embedding the projectivisation of a module?*

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[8] This argument uses that Y has finite ideal type.

3.4 Blowing up and modification

Let $X \in \mathbf{FSch}$, and let $T \trianglelefteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf. The Rees algebra

$$R_T = \bigoplus_{n\in\mathbb{N}} T^n = \bigvee_{n\in\mathbb{N}} T^n$$

is an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra generated in degree one. The Proj of the Rees algebra has principal affine charts of the form

$$\mathcal{O}_X\{T/s\} = \operatorname{colim}\left[\mathcal{O}_X \xrightarrow{s} T^k \xrightarrow{s} T^{2k} \to \cdots\right]$$

for $s \in T^k$, with the colimit taken in the category of Banach $\mathcal{O}_X$-modules. It is the universal way to make T an invertible module. It is of finite type if and only if T is finitely generated, in which case it is projective, being an affinely embedded subscheme of the projective bundle $\mathbb{P}(T)$.

3.23 Definition (Admissible modification)**.** A *blow-up* of Y *with centre* $Z_0 \subseteq Y$ is a morphism $p : \widetilde{Y} \to Y$ final among those for which $p^{-1}Z_0$ is an invertible divisor. Such is computed by taking Proj of the Rees algebra of the ideal of Z_0.

A *Z-admissible blow-up*, or blow-up *of the pair* $(Y,Z) \in {}_Z\mathbf{FSch}$ is a finite type blow-up of Y along a centre whose underlying reduced formal scheme is supported in Z. In other words, it is a Cartesian square

$$\begin{tikzcd}
(\widetilde{Y}, p^{-1}Z) \arrow[r] \arrow[d] & (\widetilde{Y}, p^{-1}Z_0) \arrow[d, "p=\mathrm{Bl}"] \\
(Y,Z) \arrow[r] & (Y,Z_0)
\end{tikzcd}$$

in ${}_Z\mathbf{FSch}$ with p the blow up along Z_0.

An *admissible modification* of (Y,Z) is a represented morphism $p : (X, p^{-1}Z) \to (Y,Z)$ such that $p^{-1}Z$ is a Cartier divisor, and which admits a factorisation

$$X \to Y' \to Y$$

with $q : Y' \to Y$ a Z-admissible blow-up along centre $Z_0 \subseteq Z$ and $X \to Y'$ a $q^{-1}Z_0$-admissible integral/projective morphism.

Blowing up is right adjoint

$$\mathrm{Bl} : {}_Z\mathbf{FSch} \to {}_Z\mathbf{FSch}^{\mathrm{inv}}$$

to the inclusion of formal schemes with invertible marking into all marked formal schemes. It is a generalisation of the right adjoint ${}_Z\mathbf{FSch}^{\mathrm{div}} \to {}_Z\mathbf{FSch}^{\mathrm{inv}}$ defined for divisorial markings in §3.2.

Stability If $f : X \to Y$, then $f^*T \twoheadrightarrow f^{-1}T\mathcal{O}_X$ and so $f^*R_T \twoheadrightarrow R_{f^{-1}T\mathcal{O}_X}$; thus the Rees algebra is functorial for morphisms of formal schemes, and moreover no relevant ideal of $R_{f^{-1}T\mathcal{O}_X}$ can pull back to the irrelevant ideal of R_T. Thus we always get a commutative square

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d, "\mathrm{Bl}_{X\times_Y Z}"'] & \widetilde{Y} \arrow[d, "\mathrm{Bl}_Z"] \\
X \arrow[r] & Y
\end{tikzcd}$$

where Z denotes the closed subscheme of Y cut out by T.

If the morphism f is flat, meaning that f^* is exact on modules, then in fact the natural map of Rees algebras is an isomorphism and hence the square is Cartesian - so in particular, the restriction to an open set of a blow-up is a blow-up.

3.24 Proposition. *Admissible blow-ups are stable for flat base change.*

The rest of this section addresses the failure of admissible blow-ups to be stable for general pullbacks or descent.

The universal property is enough to reproduce the arguments leading up to [Abb10, Prop. 3.1.17]:

3.25 Lemma. *i) A composite of admissible blow-ups is an admissible blow-up.*

ii) Let $\widetilde{Y} \to Y$ be a blow-up along a closed formal subscheme $Z \subseteq Y$, and let $X \to Y$. The blow-up of X along $X \times_Y Z$ is naturally isomorphic to the blow-up of $X \times_Y \widetilde{Y}$ along $X \times_Y \widetilde{Y} \times_Y Z$.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\mathrm{Bl}_{X\times_Y \widetilde{Y}\times_Y Z}"] \arrow[dr, "\mathrm{Bl}_{X\times_Y Z}"'] & X \times_Y \widetilde{Y} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, "\mathrm{Bl}_Z"] \\
& X \arrow[r] & Y
\end{tikzcd}$$

In particular, the saturation of the class of admissible blow-ups is stable for base change.

iii) Let $Y_\bullet \twoheadrightarrow Y$ be an open cover, $p : \tilde{Y} \to Y$ a morphism whose restriction to $Y_\bullet$ is an admissible blow-up along ideal $T_\bullet$. Then the blow-up of Y along $\prod_i T_i$ is naturally isomorphic to the blow-up of $\tilde{Y}$ along $p^* \prod_i T_i$.

$$\begin{tikzcd}
\tilde{\tilde{Y}} \arrow[r, "\mathrm{Bl}_{p^*\prod T}"] \arrow[dr, "\mathrm{Bl}_{\prod T}"'] & \tilde{Y} \arrow[d, "p"] & \tilde{Y}_\bullet \arrow[l, two heads] \arrow[d, "\mathrm{Bl}_{T_\bullet}"] \\
& Y & Y_\bullet \arrow[l, two heads]
\end{tikzcd}$$

In particular, the saturation of the class of admissible blow-ups is local on the base.

Proof. Parts *i)* and *ii)* follow as in [Abb10, 3.1.14, 17], respectively; the essential part of the argument for part *i)* can be found in [RG71, Lemme 5.1.4].

For part *iii)*, let $\tilde{\tilde{Y}}$ be the blow-up of $\tilde{Y}$ along $p^* \prod_i T_i$, where we extend T_i to Y by taking the closure. By proposition 3.24, the restriction of $\tilde{\tilde{Y}} \to \tilde{Y}$ to each $\tilde{Y}_i$ is the blow-up along $p^* \prod_i T_i$, which is the blow-up of Y_i along $\prod_i T_i$. □

Finite type blow-ups generate all projective morphisms under pullback. Indeed, let]A be a finitely generated A_0-algebra with A_0 integral over Y, and suppose, without loss of generality, that A is generated in degree one (so that $\mathscr{O}(1)$ is very ample on $\operatorname{Proj} A$). Then the blow-up of $\operatorname{Spec} A$ along the irrelevant ideal $A_+ = \bigoplus_{n>0} A_n$ is the total space $\mathbb{V}(\mathscr{O}(1))$ of the tautological line bundle on $\operatorname{Proj} A$.

Its pullback along the augmentation $A \to A_0$ is Proj of the graded algebra

$$\bigoplus_{n\in\mathbb{N}} A_+^n/A_+^{n-1} \cong \bigoplus_{n\in\mathbb{N}} A_n = A$$

that is, the zero section of $\mathbb{V}(\mathscr{O}(1))$:

$$\begin{tikzcd}
\operatorname{Proj} A \arrow[r] \arrow[d] & \mathbb{V}(\mathscr{O}(1)) \arrow[d] \\
\operatorname{Spec} A_0 \arrow[r, "0"] & \operatorname{Spec} A
\end{tikzcd}$$

If $\operatorname{Proj} A \to Y$ is Z-admissible, then $\mathbb{V}(\mathscr{O}(1)) \to \operatorname{Spec} A$ can also be written as a blow-up along $p^{-1}Z \cap 0$, where $p : \operatorname{Spec} A \to \operatorname{Spec} A_0$ is the projection.

It follows from this and part *ii)* of lemma 3.25 that the saturation of the class of Z-admissible blow-ups contains the class of finite type Z-admissible modifications.

In fact, it is possible to obtain a much more precise statement in many cases:

3.26 Theorem (Projective birational $\Rightarrow$ blow-up)**.** *Let Y be a qcqs formal scheme with an ample invertible sheaf and $Z \hookrightarrow Y$ a Cartier divisor. Let $f : \operatorname{Proj} A \to Y$ be a Z-admissible projective morphism such that $f^{-1}Z$ is invertible. Then f is a Z-admissible blow-up.*

Proof. Same proof as in [Har77, II.7.17]. □

3.27 Corollary. *The class of admissible modifications is stable under composition.*

Proof. In light of lemma 3.25, part *i)*, we only need show that a finite type blow-up $\mathrm{Bl}_Z : \widetilde{X} \to X$ of an admissible integral modification $X \to Y$ is a modification. Since Z is finitely presented, it has a model $Z_0 \hookrightarrow X_0$ under X with X_0 finite over Y. Let $\widetilde{X}_0$ denote the corresponding blow-up. By part *ii)* of lemma 3.25, $\widetilde{X} \to \widetilde{X}_0$ is the composite of the blow-up of a Cartier divisor, which is an affine embedding, and an integral morphism. □

3.28 Corollary. *Let (Y, Z) be a qcqs marked formal scheme. Let $Y_\bullet \twoheadrightarrow Y$ be an open covering, $\tilde{Y}_\bullet \to Y_\bullet$ an admissible projective modification. After possibly refining $Y_\bullet$, there exist admissible blow-ups $\tilde{\tilde{Y}} \to Y$ and $\tilde{\tilde{Y}} \times_Y Y_\bullet \to \tilde{Y}_\bullet$ making the diagram*

$$\begin{tikzcd}
\tilde{\tilde{Y}} \times_Y Y_i \arrow[r, hook] \arrow[d] & \tilde{\tilde{Y}} \arrow[dd] \\
\tilde{Y}_i \arrow[d] & \\
Y_i \arrow[r, hook] & Y
\end{tikzcd}$$

commute.

Proof. By blowing up we may assume that Z is a Cartier divisor (cf. lemma 3.25). After passing to a finite, affine refinement of $Y_\bullet$, we are in the situation of theorem 3.26, so that each member $\tilde{Y}_i$ is a blow up of Y_i along a centre $Z_i \subseteq Z \cap Y_i$. Let $\tilde{\tilde{Y}}$ be the scheme obtained by blowing up Y along the closures $\bar{Z}_i$ of the Z_i in any order. □

3.5 Finiteness for blow-ups

The purpose of this technical section is to establish a very weak form of finiteness in sufficient generality to understand the *affine* picture of rigid analytic geometry.

3.29 Lemma. *Let (Y, Z) be a divisorially marked formal $\mathbb{F}_1$-scheme, $f : \widetilde{Y} \to Y$ an admissible blow-up. Then $\widetilde{Y}$ may be supered by an admissible blow-up $f : \widetilde{Y}' \to Y$ such that $f_*\mathscr{O}_{\widetilde{Y}'}$ is a finite $\mathscr{O}_Y$-algebra.*

Proof. Any finite type admissible blow-up can be supered by a composite of admissible blow-ups along ideals generated by *two* elements. So we may assume $T = (t_1, t_2) \trianglelefteq \mathscr{O}_Y$.

Suppose we have a global function f on $\widetilde{Y}$. Its restriction f_i to the affine open subset $(t_i \neq 0)$ is a rational function $\tilde{f}_i / t_i^{n_i}$, with $\tilde{f}_i \in T^{n_i}$ an element of the Rees algebra in degree n_i. We may choose the representing quotient so that $\tilde{f}_i = c_i t_j^{n_i}$ with $j \neq i$. The existence of the global function f gives us the relation

$$\tilde{f}_1 t_2^{n_2} = \tilde{f}_2 t_1^{n_1}$$

in $T^{n_1+n_2}$.

If either n_1 or n_2 is zero, we are done. Otherwise,

$$\tilde{f}_1^{(n_1+n_2)} = c_1^{n_2}(\tilde{f}_1 t_2^{n_2})^{n_1} = c_1^{n_2}(\tilde{f}_2 t_1^{n_1})^{n_1} = c_1^{n_2} c_2^{n_1} t_1^{n_1(n_1+n_2)}$$

holds in degree $n_1(n_1+n_2)$ of the Rees algebra. That is, $c_1^{n_2} c_2^{n_1} = f_1^{n_1(n_1+n_2)}$ is in the image of $\mathscr{O}_Y$ inside $f_*\mathscr{O}_{\widetilde{Y}}$. □

4 Separation and overconvergence

We would like our definitions of separated, proper, and overconvergent morphisms to be valid over both $\mathbb{F}_1$ and $\mathbb{Z}$, and operate in a fairly uniform manner not only for schemes, formal schemes, and rigid analytic spaces, but also for the panoply of additional topoi that crop up throughout this work. It will be convenient, therefore, to allow ourselves the flexibility of working in an arbitrary coherent spatial geometric context, with a specified class $\mathbb{P}$ of morphisms satisfying the key properties obeyed by *projective* morphisms in the topos of schemes.

All of the arguments of sections 4.1, 4.2, and 4.3, which are essentially just some games you can play with commuting diagrams, are valid at this level of abstraction.

4.1 Overconvergent neighbourhoods

Let $\mathbf{S}$ be a spatial theory, $\mathbf{C} \to \mathbf{S}$ a spatial site closed under fibre products (def. 1.1). For the purposes of this paper, we may as well assume that $\mathbf{S}$ is coherent and that all representable objects are compact; however, this will not be necessary for what follows.

In this section and from now on, we fix a class of qcqs *generating overconvergent morphisms* $\mathbb{P}$ in $\mathbf{C}$, satisfying:

(P1) A composite of $\mathbb{P}$ morphisms is $\mathbb{P}$;

(P2) A base change of a $\mathbb{P}$ morphism is $\mathbb{P}$;

(P3) The diagonal of a $\mathbb{P}$ morphism is $\mathbb{P}$. *$\mathbb{P}$ morphisms are separated*

4.1 Definition. Let $U \hookrightarrow V$ be a quasi-compact open immersion. A $\mathbb{P}$-*overconvergent neighbourhood* of U/V is a factorisation $U \to \tilde{U} \to V$ such that $\tilde{U} \to V$ is $\mathbb{P}$. We will suppress the $\mathbb{P}$ from the notation if it is to be understood from the context (which it will not always be).

Let us also suppose

(P4) every overconvergent neighbourhood of U/V may be supered by one such that the inclusion of U is an open immersion;

so that after refinement, it makes sense to talk about overconvergent neighbourhoods of an overconvergent neighbourhood. In our applications, it will be possible to make this replacement *functorial*.

Overconvergent germ Since $\mathbb{P}$ is stable under composition and base change, the set of $\mathbb{P}$ morphisms with fixed target V is always cofiltered. The *overconvergent germ* of U/V is the pro-object

$$\mathrm{Sur}_{U/V} := \lim_{U \to \tilde{V} \to V} \tilde{V} \in \mathrm{Pro}(\mathbf{C}_V)$$

given as the formal limit of overconvergent neighbourhoods of of U. Since $\mathbb{P}$ is stable under base change, it extends to a presheaf

$$\mathrm{Sur}^{\mathrm{pre}}_{U/V} : \mathcal{U}_{/V} \to \mathrm{Pro}(\mathbf{C}_V)$$

of pro-objects on the small site of V.

If $U \to \tilde{V} \to V$ is an overconvergent neighbourhood of U/V, then using axiom (P4) we may define the overconvergent germ $\mathrm{Sur}_{U/\tilde{V}}$. The corresponding map

$$\mathrm{Sur}^{\mathrm{pre}}_{U/\tilde{V}} \to \mathrm{Sur}^{\mathrm{pre}}_{U/V} \tag{4}$$

is an isomorphism of presheaves.

Localising overconvergent neighbourhoods We have not required any compatibility between $\mathbb{P}$ and coverings in $\mathbf{C}$. Indeed, in our examples $\mathbb{P}$, and hence the class of $\mathbb{P}$-overconvergent neighbourhoods, will not be local on the base. In other words, the presheaf $\mathrm{Sur}^{\mathrm{pre}}_{U/V}$ is usually not a sheaf.

We may apply the plus construction to turn $\mathrm{Sur}^{\mathrm{pre}}_{U/V}$ into a sheaf $\mathrm{Sur}_{U/V} := (\mathrm{Sur}^{\mathrm{pre}}_{U/V})^+$ on V. By definition, the sections of $\mathrm{Sur}_{U/V}$ over V are the covariant functor

$$\mathrm{Hom}_S(\mathrm{Sur}_{U/V}, -) : \mathbf{S}_S \to \mathbf{Set}, \quad X \mapsto \operatorname*{colim}_{V_\bullet \twoheadrightarrow V} \operatorname*{colim}_{\tilde{V}_\bullet \xrightarrow{\mathbb{P}} V_\bullet} \mathrm{Hom}_S(\tilde{V}_\bullet, X). \tag{5}$$

The caveat is that after passing to an overconvergent neighbourhood $\tilde{V}$ of U/V, there may be new coverings available that are not pulled back from V. That is, the morphism $\mathrm{Sur}_{U/\tilde{V}} \to \mathrm{Sur}_{U/V}$ coming from (4) may no longer be an isomorphism.

This leads us to formulate the axiom of compatibility between $\mathbb{P}$ and coverings in $\mathbf{C}$:

(SC) If $f : V \to S$ is $\mathbb{P}$, then $(f_*\mathrm{Sur}^{\mathrm{pre}}_{U/V})^+ \xrightarrow{\sim} f_*\mathrm{Sur}_{U/V}$.

Thus for all V sufficiently small, the outer colimit drops out of 5, and we may assume overconvergent neighbourhoods are defined globally.

Unravelling this condition, we obtain an explicit, if slightly unwieldy, criterion:

(SC') Let $V \to S$ be $\mathbb{P}$, $U \hookrightarrow V$ a quasi-compact open immersion. Let $V_\bullet \twoheadrightarrow V$ be an open cover of V, $\tilde{V}_\bullet \to V_\bullet$ an overconvergent neighbourhood of $U_\bullet/V_\bullet$:

$$\begin{tikzcd}
& \tilde{V}_\bullet \arrow[d, "\mathbb{P}"'] \arrow[rddd, bend left] & \\
U_\bullet \arrow[ru] \arrow[r] \arrow[d, two heads] & V_\bullet \arrow[d, two heads] & \\
U \arrow[r] & V \arrow[rd, "\mathbb{P}"'] & \\
& & S
\end{tikzcd}$$

Then, after possibly passing to a cover of S, there exists an overconvergent neighbourhood of U/V whose restriction to $V_\bullet$ factors through $\tilde{V}_\bullet$.

This axiom is the only subtle part of the theory. For projective morphisms of schemes, it is a consequence of corollary 3.28.

4.2 Lemma. *The system* $\mathbb{P}$ *satisfies axiom* (SC) *if and only if for all overconvergent neighbourhoods* $\tilde{V}$ *of* U/V*,*

$$\mathrm{Sur}^+_{U/\tilde{V}} \xrightarrow{\sim} \mathrm{Sur}^+_{U/V}$$

is an isomorphism of pro-objects.

Axiom (SC) makes sense even in the absence of (P4). One can formulate a local version of the latter

(P4') every overconvergent neighbourhood of U/V may *locally* be supered by one such that the inclusion of U is an open immersion;

which, under (SC), is equivalent to (P4).

In the sequel, we will supress the superscript + from the notation of the overconvergent germ, and confuse $\mathrm{Sur}_{U/V}$ with its sections over V. Finally, we will use the term *overconvergent neighbourhood of* U/V more generally to mean any object between $\mathrm{Sur}_{U/V}$ and V.

4.2 Extension problems

Suppose we have fixed a system $\mathbb{P}$ of generating overconvergent morphisms satisfying the axioms (P1-4) and (SC). By localising, it makes sense to speak of overconvergent neighbourhoods in **S**, and not merely in the site **C**.

Suppose we are given a diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & V \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

in **S**, with $U \hookrightarrow V$ a quasi-compact open immersion. The *extension problem* is to find, locally on V, an overconvergent neighbourhood $\tilde{V}$ of U in V and an extension $\tilde{V} \to X$:

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & \tilde{V} \arrow[r] \arrow[dl, dashed] & V \arrow[d] \\
X \arrow[rr] & & S
\end{tikzcd}$$

Assuming such an extension exists, we would also like to know that it is unique up to passing to a further modification of $\tilde{V}$.

Note that an extension problem

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
X \arrow[r] & X \times_S X
\end{tikzcd}$$

is the same as a pair of extensions

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] \\
X \arrow[r] & S
\end{tikzcd}$$

in **S**, and a solution to the first problem is a proof of equality of the two solutions to the second. Thus uniqueness of extensions for a morphism is the same as existence for its diagonal.

We may also disappear S from these diagrams by working in $\mathbf{S}_S$; though note that the latter is only coherent if S is qcqs.

4.3 Definitions (Extensional properties)**.** A morphism $X \to S$ in **S** is $\mathbb{P}$-*overconvergent near* $U \to X$ if for any quasi-compact open immersion $U \hookrightarrow V$ and morphism $U \to X$ over S, there is a unique extension $\mathrm{Sur}_{U/V} \to X$ making the diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & \mathrm{Sur}_{U/V} \arrow[r] \arrow[dl, dashed] & V \arrow[d] \\
X \arrow[rr] & & S
\end{tikzcd}$$

commute. It is said to be $\mathbb{P}$-*overconvergent* if it is $\mathbb{P}$-overconvergent near every object U over X, and $\mathbb{P}$-*proper* if it is $\mathbb{P}$-overconvergent and qcqs.

We say X/S is *locally* $\mathbb{P}$-*separated* if its diagonal is $\mathbb{P}$-overconvergent. It is $\mathbb{P}$-*separated* if it is locally separated and quasi-separated, that is, if its diagonal is proper. Every $\mathbb{P}$-overconvergent morphism is locally $\mathbb{P}$-separated.

In this and the following section, we will suppress the prefix $\mathbb{P}$; however, reader beware that in later sections, this abuse of notation *will* cause confusion (def. 4.18).

Canonical extensions If U/V is an extension problem for X/S, then any solution uniquely factorises through $V \times_S X$. The latter therefore has the character of a 'canonical solution'

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \times_S X \arrow[r] \arrow[dl] & V \arrow[d] \\
X \arrow[rr] & & S
\end{tikzcd}$$

Indeed, a solution exists if and only if $\mathrm{Sur}_{U/V} \to V \times_S X$ over V. Similarly, two extensions $V \rightrightarrows X$ agree if and only if $\mathrm{Sur}_{U/V} \to V \times_{X \times X} X$ over V.

In particular:

4.4 Proposition. *Every $\mathbb{P}$ morphism is proper.*

In the situation that the replacement of (P4) can be made functorial, combining it with this construction gives a canonical solution $\tilde{V}$ with $U \hookrightarrow \tilde{V}$ a quasi-compact open immersion. Later, we will construct such solutions for formal schemes (def. 4.13).

Proper neighbourhoods Let $U \hookrightarrow V$ be a quasi-compact open immersion, and suppose that $U \hookrightarrow \tilde{V} \to V$ is a factorisation with $\tilde{V}/V$ *proper* with respect to $\mathbb{P}$. Then the definition of propriety, applied to the problem

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d, equal] \\
\tilde{V} \arrow[r] & V
\end{tikzcd}$$

implies that locally on V there is a $\mathbb{P}$ cover of $\tilde{V}$ containing U. In other words, every proper neighbourhood is an overconvergent neighbourhood.

4.5 Lemma (Enlarging $\mathbb{P}$)**.** *Let $\mathbb{P}'$ be the class of all $\mathbb{P}$-proper morphisms. Then $\mathbb{P}'$ obeys* (P1 – 3)*,* (P4')*, and* (SC)*. The theory of overconvergence for $\mathbb{P}'$ is equivalent to that for $\mathbb{P}$.*

Proof. Axioms (P1-3) follow from prop. 4.6; the remaining statements - (P4'), (SC) and that $\mathbb{P}'$-overconvergent morphisms are $\mathbb{P}$-overconvergent - follow from the fact that by definition, any $\mathbb{P}'$-overconvergent neighbourhood may locally be supered by a $\mathbb{P}$-overconvergent nehood. □

Stability A straightforward unravelling of the definitions leads to the typical stability properties (cf. [Gro60, **I**.5.5.1 & **II**.5.4.2-3]):

4.6 Proposition (Stability properties)**.** *Extensional properties of morphisms are stable under the following constructions:*

i) composition;

ii) base change;

iii) descent.

Every monomorphism is separated.

Moreover, for any composable morphisms f, g:

iv) if fg is locally separated (resp. separated), then g is locally separated (resp. separated);

v) if fg is overconvergent (resp. proper) and f is locally separated (resp. separated), then g is overconvergent (resp. proper).

Proof. I provide a proof only for the first part, by way of illustration. (The proofs of *ii)* and *iii)* are anyway addressed in example 4.12.)

Let $X \to Y \to Z$ be two overconvergent morphisms, and let U/V be an extension problem for X/Z. Overconvergence of Y/Z gives us a unique diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \mathrm{Sur}_{U/V} \arrow[r] \arrow[d, dashed] & V \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

which, by composability of $\mathrm{Sur}_{U/V}$, produces an extension problem $U/\tilde{V}$ for X/Y. Overconvergence of X/Y then gives a unique solution $\mathrm{Sur}_{U/\tilde{V}} \to X$. By lemma 4.2, $\mathrm{Sur}_{U/V} \cong \mathrm{Sur}_{U/\tilde{V}}$, whence the result. ☐

Finally, though this is not critical for much of what follows, we will often also have recourse to another axiom:

(P5) For all quasi-compact open immersions $U \hookrightarrow V$, $\mathrm{Sur}_{U/V}$ is a sheaf in the U-variable.

By quasi-compactness and locality on V, it is enough to check for quasi-compact U and hence *finite* coverings.

4.7 Proposition. *Suppose that $\mathbb{P}$ satisfies* (P5)*. Let $\{X_i \to S\}_{i\in I}$ be any family of overconvergent morphisms. Then $\mathrm{colim}_i X_i \to S$ is overconvergent.*

Proof. Let U/V be an elementary extension problem for $\mathrm{colim}_i X_i/S$. Since colimits in a topos are universal, $U_i := U \times_X X_i$ is a covering of U. ☐

4.8 Corollary. *Let $X_i \to S$ be a finite family of proper morphisms. Then $\coprod_i X_i \to S$ is proper. In particular, $\emptyset \to S$ is proper.*

4.3 Comparison principle

Suppose we have two spatial theories $\mathbf{S}_1, \mathbf{S}_2$ and a qcqs spatial geometric morphism

$$\phi : \mathbf{S}_1 \to \mathbf{S}_2.$$

It will be useful to have an understanding of how extensional properties are preserved or detected by various functors associated to ϕ. For this, we will certainly need ϕ^* to also preserve $\mathbb{P}$, so that we always get a map

$$\mathrm{Sur}_{\phi^* U/\phi^* V} \to \phi^* \mathrm{Sur}_{U/V};$$

in all our examples, this will be true by definition.

For example, although we will in [Mac15] be mainly concerned with the rigid topos Sh**Rig**, we will also want to know that for formal schemes, separation and propriety can be *detected* in the *a priori* more tractable topos Sh**FSch**.

4.9 Definition. Let $\mathbf{S}_1, \mathbf{S}_2$ be spatial theories on finitely complete sites $\mathbf{C}_1, \mathbf{C}_2$. We say that a left exact functor $F : \mathbf{C}_1 \to \mathbf{C}_2$, or an extension thereof $\mathbf{S}_1 \to \mathbf{S}_2$, *preserves* (resp. *detects*) *overconvergence* if it preserves $\mathbb{P}$ and

$$X/S \text{ overconvergent} \quad \Rightarrow \text{(resp. } \Leftarrow \text{)} \quad FX/FS \text{ overconvergent}$$

for any morphism $X \to S \in \mathbf{C}_1$. By left exactness of F, the same implication will then hold with 'overconvergent' replaced by 'locally separated'.

In our examples, $\mathbf{C}_i$ will be the site of all locally representable objects of $\mathbf{S}_i$, and F will be either the pullback or pushforward functor associated to a spatial geometric morphism between $\mathbf{S}_1$ and $\mathbf{S}_2$. It will be important to distinguish whether F preserves/detects overconvergence on all of $\mathbf{S}_1$ or merely on $\mathbf{C}_1$.

One can immediately make an elementary observation in the case F is fully faithful:

4.10 Lemma. *If F has a left exact left inverse $GF \cong 1$, and G preserves (resp. detects) overconvergence, then F detects (resp. preserves) overconvergence.*

More importantly, there is a farrago of other criteria that we will use throughout this paper.

4.11 Lemma (Comparison criteria). *Let $\phi : \mathbf{S}_1 \to \mathbf{S}_2$ be an (essential) spatial geometric morphism whose pullback preserves $\mathbb{P}$.*

- *i) Suppose that for any U/V in $\mathbf{C}_2$, any overconvergent neighbourhood of ϕ^*U/ϕ^*V can be dominated by ϕ^* applied to an overconvergent neighbourhood of U/V. That is,*

  $$\phi^*\mathrm{Sur}_{U/V} \to \mathrm{Sur}_{\phi^*U/\phi^*V} \to \phi^*V$$

  in $\mathbf{S}_1$. Then ϕ_* preserves overconvergence.

- *i') Suppose that for any U/V in $\mathbf{C}_1$, any overconvergent neighbourhood of $\phi_!U/\phi_!V$ can be dominated by $\phi_!$ applied to an overconvergent neighbourhood of U/V. That is,*

  $$\phi_!\mathrm{Sur}_{U/V} \to \mathrm{Sur}_{\phi_!U/\phi_!V} \to \phi_!V$$

  in $\mathbf{S}_2$. Then ϕ^* preserves overconvergence.

- *ii) Suppose that for any S and quasi-compact open immersion $U \hookrightarrow V$ in $\mathbf{C}_2$, the square*

  $$\begin{tikzcd}
  \mathrm{Hom}_S(\mathrm{Sur}_{U/V}, -) \arrow[r] \arrow[d] & \mathrm{Hom}_{\phi^*S}(\mathrm{Sur}_{\phi^*(U/V)}, \phi^*(-)) \arrow[d] \\
  \mathrm{Hom}_S(U, -) \arrow[r] & \mathrm{Hom}_{\phi^*S}(\phi^*U, \phi^*(-))
  \end{tikzcd}$$

  is Cartesian. Then ϕ^* detects overconvergence.

- *ii') In the situation of ii), it is enough that* $\mathrm{Hom}_S(\mathrm{Sur}_{U/V}, -)$ *surject onto the fibre product.*

iii) Suppose that $\phi_!$ preserves $\mathbb{P}$, and that for any S in $\mathbf{C}_2$ and quasi-compact open immersion U/V in $\mathbf{C}_1$, the square

$$\begin{tikzcd}
\mathrm{Hom}_S(\mathrm{Sur}_{\phi_!U/\phi_!V},-) \arrow[r] \arrow[d] & \mathrm{Hom}_{\phi^*S}(\mathrm{Sur}_{U/V},\phi^*(-)) \arrow[d] \\
\mathrm{Hom}_S(\phi_!U,-) \arrow[r] & \mathrm{Hom}_{\phi^*S}(U,\phi^*(-))
\end{tikzcd}$$

is Cartesian. Then ϕ^* preserves overconvergence.

iii') In the situation of iii), it is enough that $\mathrm{Hom}_S(\mathrm{Sur}_{\phi_!U/\phi_!V},-)$ *surject onto the fibre product.*

iv) Suppose that for any morphism $X \to S$ in $\mathbf{S}_2$ and quasi-compact open immersion U/V in $\mathbf{S}_1$, the natural map

$$\mathrm{Hom}_S(\mathrm{Sur}_{U'/V'},-) \to \mathrm{Hom}_{\phi^*S}(\mathrm{Sur}_{U/V},\phi^*(-))$$

is a colimit over the category of quasi-compact open immersions U'/V' in $\mathbf{S}_2$ equipped with a map $U/V \to \phi^*(U'/V')$. Then ϕ^* preserves overconvergence.

Proof. i) Using the adjunction property, an extension problem

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & V \arrow[d] \\
\phi_*X \arrow[r] & \phi_*S
\end{tikzcd}$$

in $\mathbf{S}_2$ transforms into a problem

$$\begin{tikzcd}
\phi^*U \arrow[r, hook] \arrow[d] & \phi^*V \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

in $\mathbf{S}_1$. If X is overconvergent, there is a unique extension $\mathrm{Sur}_{\phi^*U/\phi^*V} \to X$. The condition ensures that $\mathrm{Sur}_{\phi^*U/\phi^*V} \to \phi^*\mathrm{Sur}_{U/V}$ is an isomorphism of pro-objects. Therefore by adjunction again, $\mathrm{Sur}_{U/V} \to \phi_*X$ uniquely and ϕ_*X is overconvergent. Condition *i')* follows the from the same argument.

The remaining criteria are clear from the definition, which in general concerns the bijectivity of arrows

$$\mathrm{Hom}_S(\mathrm{Sur}_{U/V},-) \to \mathrm{Hom}_S(U,-).$$

The variants *ii'), iii')* follow by considering the diagonal. □

4.12 Example (Base change). The above principles apply to the pullback along the essential spatial geometric morphism $\phi : \mathbf{S}_{S'} \to \mathbf{S}_S$ to prove parts *ii)* and *iii)* of prop. 4.6.

ϕ^* preserves overconvergence. Apply criterion *i')* of lemma 4.11; $\phi_!$ is the functor that forgets the base morphism to S', and hence does not affect the definition of $\mathrm{Sur}_{U/V}$.

ϕ^* detects overconvergence: When $\phi : S' \twoheadrightarrow S$, then ϕ^* is comonadic, and one can show that this implies that the square in part *ii)* of 4.11 is always Cartesian.

4.4 Overconvergence in formal geometry

In the category of formal schemes, we distinguish the following four classes of morphisms.

$\mathbb{P}$	projective	$f\mathbb{P}$	formally projective
$i/\mathbb{P}$	integral/projective	$fi/\mathbb{P}$	formally integral/projective

The intersections of the classes in the second column with the Zariski site **Sch** are exactly the classes in the first column. The classes $i/\mathbb{P}$ and $fi/\mathbb{P}$ are only of passing interest, and it is possible, if a little unnatural, to entirely avoid mentioning them. The intersection of $i\mathbb{P}$ with the site of schemes of finite type over some base S is $\mathbb{P}$.

4.4.1 Elementary extension problems

Let $X \to S$ be quasi-separated, U/V quasi-compact. Then $U \to X \times_S V$ is quasi-compact, and we may pass to the embedded image $\tilde{V}$ of the affinisation of this morphism. By proposition 3.7, $U \to \tilde{V}$ is a dense, quasi-compact open immersion. The inclusion $U \hookrightarrow \tilde{V}$ can now be made affine by blowing up the reduced formal scheme with support $V \setminus U$. The resulting morphism

$$\tilde{V} = \mathrm{Bl}_{V \setminus U}(\mathrm{cl}(\mathrm{Spec}\,\mathscr{O}_V(U)/V)) \to V$$

is projective; thus $\tilde{V}$ is an overconvergent neighbourhood of U/V with respect to $f\mathbb{P}$ (or $\mathbb{P}$ if all players are schemes).

Even if X is not quasi-separated, by writing it as a filtered colimit $\mathrm{colim}_i X_i$ of quasi-separated objects in Sh**FSch**, one obtains by compactness of U a factorisation $U \to V \times_S X_i$ from which one can construct a canonical extension (depending, of course, on i).

In the special case $X = V$, the canonical solution to the extension problem is simply obtained by blowing up $\mathrm{cl}(U/V)$ along $V \setminus U$; in particular, it is $f\mathbb{P}$ over V (even $\mathbb{P}$ if V is a scheme). This naturally transforms any extension problem into one for which the inclusion is affine and dense.

4.13 Definition. Let $X \to S$ be quasi-separated, and let U/V be an extension problem. The morphism $X \leftarrow \tilde{V} \to V$ constructed above is called the *canonical extension*. The inclusion $U \hookrightarrow \tilde{V}$ to the canonical extension is an affine and (scheme-theoretically) dense open immersion (def. 3.1).

An extension problem U/V in **FSch** is called *elementary* if V is affine and $U \hookrightarrow V$ is affine and dense.

4.14 Lemma. *Every extension problem for $f\mathbb{P}$ in* Sh**FSch** *(resp. $\mathbb{P}$ in* Sh**Sch***) can be covered by elementary extension problems. That is, if U/V is any extension problem, then there exist elementary extension problems $U_\bullet/V_\bullet$ and a covering* $\mathrm{Sur}_{U_\bullet/V_\bullet} \twoheadrightarrow \mathrm{Sur}_{U/V}$.

4.4.2 Overconvergence for schemes and formal schemes

In cases of interest, the classes $\mathbb{P}, i/\mathbb{P}, f\mathbb{P}, fi/\mathbb{P}$ all define the same notion (def. 4.18) of overconvergence.

4.15 Lemma. *The classes $\mathbb{P}$ and $i/\mathbb{P}$ in* **Sch** *and $f\mathbb{P}$ and $fi/\mathbb{P}$ in* **FSch** *obey the axioms* (P1-5) *and* (SC).

Proof. Axioms (P1-3) are handled by proposition 3.17, while (P4) follows from the construction of canonical solutions for formally projective morphisms (which are, in particular, quasi-separated.

The locality axiom (SC) is a consequence of corollary 3.28.

For (P5), over $\mathbb{F}_1$, after reducing the question to an elementary extension problem, the covering condition is trivial and so there is nothing to check. For $\mathbb{Z}$, the result is much harder and relies on equating our approach to propriety with the classical one (thm. 4.39). Assuming this, one may prove it as follows.[9]

Let U/V be an elementary extension problem, $U_\bullet \twoheadrightarrow U$ a covering, $\mathrm{Sur}_{U_\bullet/V} \to X$ a morphism. We may assume X is quasi-separated. For each U_i, the extension $\tilde{V}_i = \mathrm{cl}(U_i/V \times X)$ is proper over V. Since closure commutes with finite unions, we have a surjective map

$$\coprod_i \tilde{V}_i \to \tilde{V}$$

to the extension $\tilde{V} = \mathrm{cl}(U/V \times X) \to X$. By [Gro60, **II**.5.2.3.*ii)*], $\tilde{V}$ is proper over V. Thus $\mathrm{Sur}_{U/V} \to X$. □

4.16 Aside (Why formally projective?). The conclusion of lemma 4.15 is false for the class $\mathbb{P}$ in **FSch**; it is essential to allow *formally* projective, rather than simply projective, modifications. For instance, the proof of axiom (P4) relies on passing to the embedded closure of an open immersion $U \hookrightarrow V$ - which in general fails to be representable by schemes over V.

More importantly, the ability to also take point-set-topological closures within $\mathbb{P}$ is a necessary condition for (SC). Indeed, this axiom requires that for sufficiently small V and any open immersion $V' \hookrightarrow V$ disjoint from U, we should be able to find a modification of V whose pullback to V' factors through the overconvergent neighbourhood $\emptyset$ of $U \cap V'/V'$.

4.17 Proposition. *Let f be a locally finite type morphism in* Sh**FSch**. *The following are equivalent:*

i) f is $fi/\mathbb{P}$-overconvergent in Sh**FSch**;

ii) f is $f\mathbb{P}$-overconvergent in Sh**FSch**.

To establish overconvergence of f, it is enough to exhibit solutions to extension problems U/V with V a local formal scheme.

Suppose that f in fact lies in the subcategory Sh**Sch**. *Then the above are moreover equivalent to the following:*

iii) f is $i/\mathbb{P}$-overconvergent in Sh**Sch**;

iv) f is $\mathbb{P}$-overconvergent in Sh**Sch**.

To establish overconvergence of f, it is enough to exhibit solutions to extension problems U/V with V a local scheme.

In light of proposition 4.17, the definitions below are unambiguous. The proof and discussion of this fact occupies the rest of this section (4.4).

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[9] For the reader concerned about circular reasoning: we will not use axiom (P5) until the sequel [Mac15].

4.18 Definitions. A morphism of formal schemes is said to be *overconvergent* (resp. *proper*) if it is locally of finite type (resp. and quasi-compact) and satisfies the equivalent conditions *i), ii)* of proposition 4.17. A morphism is *locally separated* (resp. *separated*) if its diagonal is overconvergent (resp. proper).

A morphism of schemes is overconvergent (resp. proper, locally separated, separated) if it is so when considered as a morphism of formal schemes; equivalently, if it satisfies the equivalent conditions *iii), iv)* of 4.17.

Proof of 4.17. The implications *ii)⇒i)* and *iv)⇒iii)* are the easy directions, by the inclusions $\mathbb{P} \subset i/\mathbb{P}$ and $f\mathbb{P} \subset fi/\mathbb{P}$. The reverse implications, as well fact that these properties may be checked on local objects, follow from a standard compactness argument; for more discussion, see §4.4.4 below.

Let us focus on the two possible definitions for schemes. In the language of §4.3, we want to show that the inclusion

$$(\phi^* \mathrm{Sh}\mathbf{Sch}, \mathbb{P}) \hookrightarrow (\mathrm{Sh}\mathbf{FSch}, f\mathbb{P})$$

detects and preserves overconvergence.

To show *i/ii)⇒iii/iv)*, it will be enough to show that the pushforward $\phi_* : \mathrm{Sh}\mathbf{FSch} \to \mathrm{Sh}\mathbf{Sch}$ preserves overconvergence (cf. lemma 4.10). For this, we will use the comparison criterion *i)* of lemma 4.11.

Let U/V be an affine open immersion in **Sch**. We want to know that the overconvergent germ $\mathrm{Sur}_{U/V}$ does not depend on whether we consider U/V as objects of **Sch** or of **FSch**. In other words, we must show that every $f\mathbb{P}$-overconvergent neighbourhood $\tilde{V}$ of U/V can be dominated by a $\mathbb{P}$-overconvergent neighbourhood. It is enough to take the embedded closure of the affinisation of U over $\tilde{V}$; this will always be a scheme and hence projective over V.

4.19 Corollary. *The Yoneda embedding* $\mathbf{FSch} \hookrightarrow \mathrm{Sh}\mathbf{Sch}$ *preserves overconvergence.*

As for the converse statement, we must check that a $\mathbb{P}$-overconvergent scheme automatically has solutions to all extension problems with U/V arbitrary *formal* schemes. We will achieve this by showing that any such extension problem over a morphism $X \to S$ in $\mathrm{Sh}\mathbf{Sch}$ can be factored through an open immersion U'/V' of schemes over X/S; this is criterion *iv)* of lemma 4.11.

4.20 Lemma. *Let* $A \to A[f^{-1}]$ *be a localisation,* $B_f \to A[f^{-1}]$ *a ring homomorphism,* $\tilde{f} \in B_f$ *a unit lifting* f*. Then* $B := B_f \times_{A[f^{-1}]} A \to B_f$ *is a localisation at* $\tilde{f}$*, and* $A[f^{-1}] \cong A \otimes_B B_f$*.*

Proof. By commutativity of finite limits with filtered colimits, and of localisation with base change, respectively. □

Let X/S be a morphism in $\mathrm{Sh}\mathbf{Sch}$, U/V an elementary extension problem for X/S in **FSch**. Suppose U is defined by the non-vanishing of $f \in \mathcal{O}(V)$. We will construct an extension problem U'/V' in **Sch** and a Cartesian square

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
U' \arrow[r] & V'
\end{tikzcd}$$

for which purpose we may replace X and S with affine open subsets.

Let $U' := \mathbb{G}_{m,X}$. The invertible function f on U defines a morphism $U \to U'$. Define V' by the exactness of the pullback square

$$\begin{tikzcd}
\mathscr{O}(V') \arrow[r] \arrow[d] & \mathscr{O}(X)[f^{\pm 1}] \arrow[d] \\
\mathscr{O}(V) \arrow[r] & \mathscr{O}(U)
\end{tikzcd}$$

By lemma 4.20 applied to the discrete quotients of $\mathscr{O}(V)$, $\mathscr{O}(V') \to \mathscr{O}(U')$ is a localisation. Therefore *iii / iv)⇒i / ii)*. □

4.21 Aside (Non-quasi-compact extension problems). For schemes, it is possible to reduce the definition of quasi-compactness (and therefore, by consideration of the diagonal, quasi-separatedness) to a certain non-quasi-compact extension problem.

Let $X = \bigcup_i U_i$ be a scheme written as a union of affine open subsets. Taking the product of $\mathscr{O}(U_i)$ in the category of discrete algebras, we obtain an open immersion

$$\coprod_i U_i \hookrightarrow \operatorname{Spec} \prod_i \mathscr{O}(U_i).$$

A solution to the associated extension problem is a diagram

$$\begin{tikzcd}
& U' \arrow[d] \arrow[dr, "f"] & \\
\coprod_i U_i \arrow[ur, hook] \arrow[r, hook] & \operatorname{Spec} \prod_i \mathscr{O}(U_i) & X
\end{tikzcd}$$

with the vertical arrow projective, and so in particular quasi-compact. Since the target is affine, we may assume that U' is covered by the pullbacks of finitely many U_is along f. This finite list of affine sets covers each member of the cover $\{U_i \subseteq X\}$ and therefore X itself.

The converse appears to require a good theory of quasi-affine morphisms, which is a little delicate in the non-quasi-compact case. Nonetheless, using [Sta14, 01P9] it is possible to prove that the following are equivalent for a scheme X/S:

i) all extension problems $\iota : U \hookrightarrow V$ such that $\iota_* \mathscr{O}_U$ is $\mathscr{O}_V$-quasi-coherent have unique solutions;

ii) $X \to S$ is $\mathbb{P}$-proper.

Since as it stands the general criterion is a little clumsy, I omit it from the development.

4.4.3 Expansions

Let U/V be an affine extension problem for formal schemes. We also suppose given a function $f \in \mathscr{O}(V)$ whose non-vanishing defines U. Let $Z \subseteq V$ be a finitely presented closed subscheme in which $Z_U := Z \cap U$ is dense. We do not assume U is dense in V.

We define an operation called *intermediate expansion* on the data (U, V, Z), as follows: let $p : \tilde{V} \to V$ be the blow-up along $Z \cap (f = 0)$, and write $\tilde{U} = \tilde{V} \setminus \left(p_*^{-1}(f = 0)\right)$ where p_*^{-1}

denotes the operation of strict transform along the blow-up p [Sta14, 080D]. Explicitly, if T is the ideal defining Z, then

$$\tilde{U} = \operatorname{Spec} \mathscr{O}_V\{T/f\}$$

and the ideal defining $p_*^{-1}Z \cap \tilde{U}$ is $T/f \trianglelefteq \mathscr{O}_V\{T/f\}$. We get a commutative diagram

$$\begin{tikzcd}
\tilde{U} \arrow[r] & \tilde{V} \arrow[d] & p_*^{-1}Z \arrow[l] \arrow[d, equal] \\
U \arrow[u] \arrow[r] & V & Z \arrow[l]
\end{tikzcd}$$

of formal schemes over S, with $U \hookrightarrow \tilde{U}$ an open immersion. Since the intersection of the discriminant locus with Z is a Cartier divisor, the restriction $p_*^{-1}Z \to Z$ is an isomorphism. It follows that $Z_{\tilde{U}} := Z \cap \tilde{U}$ remains dense in Z.

If U is representable by schemes over some base formal scheme S, then so is the affine part $\tilde{U}$ of the intermediate expansion:

4.22 Lemma. *Let $V \to S$, $U \hookrightarrow V$ an affine, dense open immersion whose complement is defined by an equation $(f = 0)$. Suppose that U is representable by schemes over S. Let $I_V \trianglelefteq \mathscr{O}_V$ be an ideal of definition. Then* $\operatorname{Spec} \mathscr{O}_V\{I_V/f\}$ *is representable by schemes over S.*

Proof. The crux will be to show that $\mathscr{O}_V\{I_V/f\}$ is a Banach module over $\mathscr{O}_S$. Let I_S be an ideal of definition for $\mathscr{O}_S$, and $I_{V/S}$ the image of I_V in $\mathscr{O}_V/I_S$. We will show that $I_{V/S}$ is nilpotent in $\mathscr{O}_V\{I_V/f\}/I_S$ and hence that the latter is discrete.

Since by hypothesis $\mathscr{O}_V\{f^{-1}\}$ is Banach over $\mathscr{O}_S$, there is some $k \in \mathbb{N}$ for which $f^k I_{V/S}$ is nilpotent in $\mathscr{O}_V/I_S$. Thus

$$I_{V/S}^{k+1} = f^{k+1}(I_{V/S}/f)^{k+1} \subseteq f^{k+1}(I_{V/S}/f) = f^k I_{V/S}$$

is nilpotent in $\mathscr{O}_V\{I_V/f\}/I_S$. □

By iterating intermediate expansions, there are some inductive constructions that are useful as technical tools in the sequel.

4.23 Definition (Expansions)**.** Write $(U_0, V_0, Z_0) = (U, V, Z)$.

If we set $(U_{i+1}, V_{i+1}, Z_{i+1}) = (\tilde{U}_i, \tilde{V}_i, Z_i \times_{V_i} \tilde{V}_i)$, then the formal V-scheme

$$U^{\text{él}} := \operatorname*{colim}_{i \to \infty} U_i$$

is called the *expanded degeneration* of $U \subseteq V$.

If instead we write $(U_{i+1}, V_{i+1}, Z_{i+1}) = (U_0, \tilde{V}_i, \tilde{Z}_i)$, then instead we get a projective system

$$\operatorname{Sur}_{U/V}^Z := \lim_{i \to \infty} V_i$$

defined as a pro-object of $\mathbf{FSch}_S$. It is the largest quotient of $\operatorname{Sur}_{U/V}$ admitting a section over Z. Now writing $(U_{i+1}, V_{i+1}, Z_{i+1}) = (U_0, \tilde{U}_i, \tilde{Z}_i)$, we get a pro-open immersion

$$\operatorname{sur}_{U/V}^Z := \lim_{i \to \infty} U_i \hookrightarrow \operatorname{Sur}_{U/V}^Z.$$

In fact, since the transition maps of this one are affine, $\operatorname{sur}_{U/V}^Z$ even has a limit as a formal scheme, affine and representable by schemes over V. It is usually not of finite type (or Noetherian), and so for our purposes it will be useful to anyway consider $\operatorname{sur}_{U/V}^Z$ as a pro-object.

4.24 Example. The universal cover of the Tate formal curve over a complete DVR $\mathcal{O}_K$ is an example of a $U^{\text{él}}$, with initial data $V = \mathbb{P}^1_{\mathcal{O}_K}, U = V \setminus \{0,\infty\}, Z = \mathbb{P}^1_k$.

4.25 Aside. Later, we will introduce certain topological objects parametrising toric formal schemes. The above constructions should be thought of in terms of the following analogies on the (cone over the) unit interval $[0,1]$:

U/V	$[0,0]/[0,1]$
$U^{\text{él}}$	$[0,1)$
$\mathrm{Sur}^Z_{U/V}$	$[0,1]$ subdivided at points $2^{-k}, k \in \mathbb{N}$
$\mathrm{sur}^Z_{U/V}$	$\lim_{k\to\infty}[0,2^{-k}]$

This analogy can be made precise using rigid analytic geometry [Mac15].

Expanded degeneration The purpose of the construction $U/V \mapsto U^{\text{él}}/V$ is to replace a quasi-compact open immersion with an overconvergent morphism.

4.26 Lemma. *Let U/V be an affine open immersion of formal schemes, $Z \hookrightarrow V$ a closed subscheme that set-theoretically contains U (for example, one that contains an ideal of definition). Then the formally embedded closure $\mathrm{cl}(U/\tilde{V})$ of U in the intermediate expansion $\tilde{V}$ is contained in $\tilde{U}$.*

Proof. This is a consequence of the elementary fact that a blow-up of the intersection of two subschemes separates the strict transforms of those subschemes. ☐

4.27 Proposition. *Let U/V be an affine open immersion of formal schemes, $Z \hookrightarrow V$ a closed subscheme that set-theoretically contains U. Then $U^{\text{él}} \to V$ is an overconvergent morphism.*

Proof. First note that by lemma 4.22, $U^{\text{él}} \to V$ is locally of finite type. Let U/V be an elementary extension problem. Then U factors through some finite stage $U_i \subseteq V_i$ of the expanded degeneration. By functoriality of formally embedded closures, $V \times_{V_i} V_{i+1}$ then factors through $\mathrm{cl}(U_i/V_{i+1}) \subseteq U_{i+1}$. ☐

***Z*-rational overconvergent germ** The second and third constructions of definition 4.23 provide certain canonical covers in the category of formal algebraic spaces.

4.28 Lemma. *Suppose that U is dense in V. The square*

$$\begin{tikzcd}
Z_U \arrow[r] \arrow[d] & Z \arrow[d] \\
U \arrow[r] & \lim \mathrm{sur}^Z_{U/V}
\end{tikzcd}$$

is a pushout in the category of affine formal schemes.

Proof. Assume $V = \mathrm{Spec}\, A$ is affine. We are looking at morphisms

$$\begin{tikzcd}
 & A[f^{-1}] \arrow[d, two heads] \\
A/I \arrow[r, hook] & A/I[f^{-1}]
\end{tikzcd}$$

and I'm claiming that the fibre product is just $\bigcup_{k\to\infty} A[I/f^k] \subseteq A[f^{-1}]$. Certainly this injects into the fibre product, so it will suffice to produce a section. If $g/f^k \in A[f^{-1}]$ has image in A/I, then it can be written in the form $h_1 + h_2/f^k$ with $h_1 \in A$ and $h_2 \in I$. □

4.29 Proposition. *Let U/V be an elementary extension problem over S, and suppose that V is local. The square*

$$\begin{tikzcd} Z_U \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & \mathrm{sur}^Z_{U/V} \end{tikzcd}$$

is a pushout over the category of formal algebraic spaces locally of finite type over S.

Proof. Indeed, the global statement follows from the fact that Z is local, and so U must factor through any affine open subset of X containing the image of the closed point of Z. □

4.4.4 Finiteness

In this section we'll take a closer look at the statements in proposition 4.17 that depend essentially on the finiteness of f.

4.30 Proposition. *Let $f : X \to S$ be a morphism in* Sh**FSch** *or* Sh**Sch***, locally of finite type (resp. presentation). To establish overconvergence of f, it is enough to exhibit solutions to extension problems U/V either*

i) for all V of finite type (resp. presentation) over S; or

ii) for all local V essentially of finite type (resp. presentation) over S.

Over $\mathbb{F}_1$, one can check with V local and of finite type (resp. presentation) over S.

Proof. For part *i)*, it is equivalent to check that for any S, the inclusions

$$\mathrm{Sh}\mathbf{FSch}_S^{\mathrm{lpf}} \hookrightarrow \mathrm{Sh}\mathbf{FSch}_S^{\mathrm{ltf}} \hookrightarrow \mathrm{Sh}\mathbf{FSch}_S \tag{6}$$

$$\mathrm{Sh}\mathbf{Sch}_S^{\mathrm{lpf}} \hookrightarrow \mathrm{Sh}\mathbf{Sch}_S^{\mathrm{ltf}} \hookrightarrow \mathrm{Sh}\mathbf{Sch}_S \tag{7}$$

of the topoi of (formal) schemes locally of finite presentation, resp. type, over S preserves overconvergence. If we consider the right-hand topoi to be endowed with classes $fi/\mathbb{P}, i/\mathbb{P}$, respectively, then this argument will also fill in the details of the proof of proposition 4.17.

It is also worth mentioning that the topoi of finitely presented objects do not obviously satisfy (P4), as a canonical solution (def. 4.13) need not be finitely presented. The rôle of these topoi is sufficiently auxiliary that this will not cause us any serious problems.

4.31 Lemma. *If X/S is locally of finite type (resp. presentation), then for any quasi-compact extension problem U/V there exists a finite type (resp presentation) extension problem U'/V' over S such that $U' \to X$ factors $U \to X$.*

Proof. I provide the argument for finite type. Let U/V be an elementary extension problem, and write $\mathcal{O}_S(V)$ as a filtered union $\mathcal{O}_S(V_i)$ of $\mathcal{O}_S$-algebras of finite type over which U is defined. Then $\mathcal{O}_S(U) = \bigcup_i \mathcal{O}_S(U_i)$ is also a filtered union.

Replacing X with an affine subset through which U factors, finiteness implies that

$$\mathscr{O}_S(X) \to \mathscr{O}_S(U) \cong \bigcup_i \mathscr{O}_S(U_i)$$

factors through $\mathscr{O}_S(U_i)$ for some i. $\square$

This gives us the first part of the result by part *iv)* of lemma 4.11.

The second part is essentially a repetition of the same finiteness argument. Suppose that f has unique solutions to all extension problems for local objects, and let U/V be an extension problem. For each $p \in V$, let $V_p = \lim_{f(p)\neq 0} V_f$ denote the local scheme of V at p, $U_p := U \times_V V_p$. By hypothesis, there exists a formally projective modification $\tilde{V}_p \to V_p$ such that $\tilde{V}_p \to X$ under U.

By definition and by lemma 3.11, this modification can be written as a finite morphism, followed by a formal completion (along a finitely presented subscheme), followed by a projective morphism. By an affine embedding, it can be replaced with one whose finite and projective parts are finitely *presented*. This replacement $\tilde{V}_p$ has a formally projective model over V_f for a coinitial family of V_f. Moreover, one can take a standard affine atlas

$$\tilde{V}_f = \bigcup_{j=1}^n \tilde{V}_f^j, \quad \tilde{V}_p^j \cong \lim_{f(p)\neq 0} \tilde{V}_f^j$$

indexed by a finite set independent of f. Write $U_f^j := U \times_V \tilde{V}_f^j$.

Let $X^j \subseteq X$ be an affine open subset containing the image of $\tilde{V}_p^j$, and let $\mathscr{O}_S\{\vec{x}\} \twoheadrightarrow \mathscr{O}_S(X^j)$ be a presentation by finitely many generators. By cocompactness, there is a homomorphism $\mathscr{O}_S\{\vec{x}\} \to \mathscr{O}_S(\tilde{V}_f^j)$ making the square

$$\begin{tikzcd}
\mathscr{O}_S\{\vec{x}\} \arrow[r, two heads] \arrow[d] & \mathscr{O}_S(X^j) \arrow[d] \\
\mathscr{O}_S(\tilde{V}_f^j) \arrow[r, hook] & \mathscr{O}_S(U_f^j)
\end{tikzcd}$$

commutative. It follows that $\mathscr{O}_S(X^j) \to \mathscr{O}_S(\tilde{V}_f^j)$. By varying j from 1 to n, we obtain an f such that $\tilde{V}_p \to X$ extends to $\tilde{V}_f$.

Finally, by varying p we obtain a covering of V and hence a solution $\mathrm{Sur}_{U/V} \to X$. $\square$

There is also the question of whether the inclusions 6, 7 *detect* overconvergence, and as such whether the theory of overconvergence can be formulated entirely in terms of the category of objects of finite type (resp. presentation) over S. For the inclusion

$$\mathrm{Sh}\mathbf{Sch}_S^{\mathrm{ltf}} \to \mathrm{Sh}\mathbf{Sch}_S,$$

the affirmative answer is a consequence of the fact that every $\mathbb{P}$ modification of a scheme V/S of finite type remains of finite type.

However, as remarked above in 4.16, the inclusion

$$\mathrm{Sh}\mathbf{FSch}_S^{\mathrm{ltf}} \to \mathrm{Sh}\mathbf{FSch}_S$$

certainly does not detect overconvergence: not every extension problem U/V for an overconvergent morphism X/S of formal schemes can be solved by a modification representable by schemes over V. One way to fix this would be to weaken the finiteness condition to *formally* of finite type over S, which is satisfied by morphisms in $f\mathbb{P}$. We will not pursue this approach here.

4.32 Aside. Let $\tilde{V}$ be any overconvergent neighbourhood of U/V. Applying the intermediate expansion (§4.4.3) to the data $U, \tilde{V}, Z$ with $Z \subseteq \tilde{V}$ the closed subscheme cut out by an ideal of definition does provide an overconvergent neighbourhood $\tilde{U}$ of U/V which is affine and, by lemma 4.22, of finite type over V. Indeed, this blow-up separates Z from $V \setminus U$, and so the formal completion of $\tilde{U}$ along the strict transform $p_*^{-1}Z$ is formally projective over V.

Extending this to the construction $U^{\text{él}}$ of def. 4.23, one can even arrange that $U^{\text{él}} \to V$ is an *overconvergent morphism*, as can be seen, at least in the Noetherian case, from the valuative criterion (4.48). I omit the details. This is analogous (and, as we shall see in [Mac15], directly related) to the ability to refine a neighbourhood to an *open* neighbourhood in general topology.

We may therefore replace the system $f\mathbb{P}$ with the set of overconvergent morphisms (lemma 4.5), thereby defining - somewhat circularly - a reasonable theory of overconvergent neighbourhoods in the topos $\mathrm{Sh}\mathbf{FSch}_S^{\mathrm{ltf}}$.

4.5 Reduction to the underlying space

As the usual definitions of separated and proper morphisms from algebraic geometry lead us to expect (cf. [Gro60, **I**.5.5.1.vi, **II**.5.4.6]), extensional properties of a scheme can be understood at the level of its underlying reduced scheme.

4.33 Proposition. *A morphism $f : X \to S$ in* **FSch** *is overconvergent if and only if its reduction $X^{\mathrm{red}} \to S^{\mathrm{red}}$ is overconvergent.*

To establish overconvergence of f, it is enough to exhibit solutions to extension problems U/V with V a reduced scheme - which may be taken of finite type over S, or local and essentially of finite type over S.

Proof. Let us denote by

$$\mathrm{dR}_! : \mathbf{Sch}^{\mathrm{red}} \leftrightarrows \mathbf{FSch} : \mathrm{dR}^*$$

the inclusion and reduction functors betweeen the category of formal schemes (having locally an ideal of definition) and its full, coreflective subcategory $\mathbf{Sch}^{\mathrm{red}}$ of reduced schemes. They extend to an essential geometric morphism

$$\mathrm{dR} : \mathrm{Sh}\mathbf{Sch}^{\mathrm{red}} \to \mathrm{Sh}\mathbf{FSch}.^{10}$$

The theorem has two statements. The first is that if $X \in \mathbf{FSch}_S$ is overconvergent, then its reduction $\mathrm{dR}_!\mathrm{dR}^*X$ is overconvergent. The second is that X is overconvergent as soon as dR^*X is overconvergent. Both are subsumed, in the language of §4.3, by the lemma:

4.34 Lemma. *The functors*

$$\mathrm{dR}^* : \mathbf{FSch} \to \mathbf{Sch}^{\mathrm{red}}, \quad \mathrm{dR}_! : \mathrm{Sh}\mathbf{Sch}^{\mathrm{red}} \to \mathrm{Sh}\mathbf{FSch}$$

---

[10] So named because $\mathrm{dR}_*\mathrm{dR}^*X$ on a scheme X is often called the 'de Rham sheaf' and denoted X^{dR}.

detect overconvergence. Their adjoints

$$\mathrm{dR}_! : \mathbf{Sch}^{\mathrm{red}} \to \mathbf{FSch}, \quad \mathrm{dR}^* : \mathrm{Sh}\mathbf{FSch} \to \mathrm{Sh}\mathbf{Sch}^{\mathrm{red}}$$

preserve overconvergence.

Proof. In light of the fact that dR^* is a left inverse to $\mathrm{dR}_!$, it will suffice to prove the statements for the former; cf. lemma 4.10.

Let us begin with the second statement. Let $U \hookrightarrow V$ be a quasi-compact open immersion in $\mathbf{Sch}^{\mathrm{red}}$ and take an overconvergent neighbourhood $\tilde{V}$ of U/V in $\mathbf{FSch}$. The reduction $\tilde{V}^{\mathrm{red}} \to \tilde{V}$ is embedded and contains U, thus in particular an overconvergent neighbourhood of U/V. Thus

$$\mathrm{Sur}_{\mathrm{dR}_!(U/V)} \cong \mathrm{dR}_! \mathrm{Sur}_{U/V},$$

and dR^* preserves overconvergence.

It remains only to show that dR^* detects overconvergence. At this point, I should clarify that, more precisely, we are investigating

$$\mathrm{dR}^* : (\mathbf{FSch}, fi/\mathbb{P}) \to (\mathbf{Sch}^{\mathrm{red}}, i/\mathbb{P}).$$

By criterion *iii)* of lemma 4.11, we have to show that for $X \to S$ and a quasi-compact open immersion $U \hookrightarrow V$ in $\mathbf{FSch}$, the square

$$\begin{tikzcd}
\mathrm{Hom}_S(\mathrm{Sur}_{U/V}, -) \arrow[r] \arrow[d] & \mathrm{Hom}_{S^{\mathrm{red}}}(\mathrm{Sur}_{(U/V)^{\mathrm{red}}}, X^{\mathrm{red}}) \arrow[r, equal] \arrow[d] & \mathrm{Hom}_S(\mathrm{Sur}_{(U/V)^{\mathrm{red}}}, X) \arrow[d] \\
\mathrm{Hom}_S(U, -) \arrow[r] & \mathrm{Hom}_{S^{\mathrm{red}}}(U^{\mathrm{red}}, X^{\mathrm{red}}) \arrow[r, equal] & \mathrm{Hom}_S(U^{\mathrm{red}}, X)
\end{tikzcd}$$

is Cartesian. This is handled by lemma 4.35. □

4.35 Lemma (Independence of nilpotents)**.** *If U/V is quasi-compact and $V_0 \to V$ is a nilpotent embedding of schemes, then*

$$\begin{tikzcd}
\mathrm{Hom}_S(\mathrm{Sur}_{U/V}, X) \arrow[r] \arrow[d] & \mathrm{Hom}_S(\mathrm{Sur}_{U_0/V_0}, X) \arrow[d] \\
\mathrm{Hom}_S(U, X) \arrow[r] & \mathrm{Hom}_S(U_0, X)
\end{tikzcd}$$

is Cartesian. That is, solutions of an extension problem $U \to X$ correspond to those of the restricted problem $U_0 \to X$.

Proof. Apply lemma 4.28 to the data $U, V, Z = V_0$ to see that

$$\begin{tikzcd}
U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\
U \arrow[r] & \lim \mathrm{sur}_{U/V}^{V_0}
\end{tikzcd}$$

is a pushout in the category of algebraic spaces. The blow-up of the intersection of a nilpotent ideal with a Cartier divisor is finite; therefore $\mathrm{Sur}_{U/V}^{Z} = \mathrm{sur}_{U/V}^{Z}$ is pro-finite over V. Thus $\lim \mathrm{sur}_{U/V}^{Z} \to V$ is integral and so $\mathrm{Sur}_{U/V} \to \lim \mathrm{sur}_{U/V}^{Z}$ with respect to $i/\mathbb{P}$. □

Finally, to see that overconvergence can still be checked using reduced *and* finite type (or local and essentially finite type) test spaces, it is enough to observe that if a formal scheme is of finite type, resp. local, then so is its reduction. ☐

4.36 Corollary. *The Yoneda embedding* $\mathbf{FSch} \to \mathrm{Sh}\mathbf{Sch}$ *detects overconvergence.*

Beware that strictly speaking this Yoneda functor does not detect *quasi-compactness*; fortunately, this will not usually cause confusion.

4.6 Base change $\mathbb{F}_1 \to \mathbb{Z}$

For our definition of properness of $\mathbb{F}_1$-schemes to say anything useful about ordinary algebraic geometry, we must have two things: first, that it is preserved by the base change functor to ordinary schemes over $\mathbb{Z}$, and second, that over $\mathbb{Z}$ our definitions are equivalent to the ones in [Gro60] that all know and love.

4.37 Proposition. *The base change* $p^* : \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1} \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{Z}}$ *preserves overconvergence.*

Proof. We will use criterion *i')* of lemma 4.11, which we note does not require the forgetful functor $p_!$ to preserve $\mathbb{P}$ or even to be defined on the whole of $\mathrm{Sh}\mathbf{Sch}_{\mathbb{Z}}$. Indeed, one can still define $p_!\mathrm{Sur}_{U/V}$ for affine U/V as a left pro-adjoint

$$\mathrm{Hom}(p_!\mathrm{Sur}_{U/V}, -) := \mathrm{Hom}(\mathrm{Sur}_{U/V}, p^*(-))$$

to p^*, whence our objective is simply to find a section to the natural map

$$\mathrm{Hom}(\mathrm{Sur}_{U/V}, p^*(-)) \to \mathrm{Hom}(\mathrm{Sur}_{p_!U/p_!V}, -).$$

Let U/V be an elementary extension problem for schemes over $\mathbb{Z}$, and let $p_!U/p_!V$ be the open immersion of affine $\mathbb{F}_1$-schemes obtained by forgetting the additive structure. (The assumption that U/V are affine is essential here for $p_!$ to be defined.) By theorem 3.26, after refinement, any overconvergent neighbourhood $\tilde{V}_{\mathbb{F}_1}$ of $p_!U/p_!V$ is a blow-up along some finitely generated ideal $T \trianglelefteq \mathscr{O}(p_!V)$.

Let $\mathbb{Z}T \trianglelefteq \mathscr{O}(V)$ denote the additive closure of T, $\tilde{V}_{\mathbb{Z}}$ the blow-up of V along $\mathbb{Z}T$. Since T generates $\mathbb{Z}T$, the morphism of Rees algebras $\bigoplus_i T^i \otimes_{\mathbb{F}_1} \mathbb{Z} \to \bigoplus_i \mathbb{Z}T^i$ induces a morphism $p_!\tilde{V}_{\mathbb{Z}} \to \tilde{V}_{\mathbb{F}_1}$ over $p_!V$.

$$\begin{tikzcd}
 & \tilde{V}_{\mathbb{Z}} \arrow[dd, "\mathrm{Bl}_{\mathbb{Z}T}"] & & & p_!\tilde{V}_{\mathbb{Z}} \arrow[r] & \tilde{V}_{\mathbb{F}_1} \arrow[dd, "\mathrm{Bl}_T"] \\
 & & {} \arrow[r, dotted, "p_!"] & {} & & \\
U \arrow[uur] \arrow[r] & V & & & p_!U \arrow[uu] \arrow[r] & p_!V
\end{tikzcd}$$

Thus $p_!\mathrm{Sur}_{U/V} \to \mathrm{Sur}_{p_!U/p_!V}$ over $p_!V$. ☐

It is not possible to repeat this argument directly for formal schemes due to the absence of the forgetful functor $p_!$ even for affine objects (as remarked at the end of §2.3). However, we can to deduce its conclusion from the fact that overconvergence depends only on the underlying reduced scheme.

4.38 Corollary. *The base change* $p^* : \mathbf{FSch}_{\mathbb{F}_1} \to \mathbf{FSch}_{\mathbb{Z}}$ *preserves overconvergence.*

Proof. Follows from 4.19, 4.36, and 4.37. ☐

4.39 Proposition. *A finite type, separated morphism in* $\mathbf{FSch}_{\mathbb{Z}}$ *is proper if and only if it is universally closed.*

Proof. The reductions of corollary 4.33 allow us to assume that X/S are (reduced) schemes. Suppose that X/S is proper. The Chow property then implies that X is dominated by a projective S-scheme, and is therefore universally closed by [Gro60, **II**.5.2.3.*ii)*].

The difficulty lies in showing that if $X \to S$ is proper in the sense of [Gro60, **II**.5], then it is proper in the sense of definition 4.18. Let U/V be a quasi-compact extension problem, and let $\tilde{V} \to X$ be the canonical extension (def. 4.13). Then $\tilde{V} \to V$ is also of finite type, separated, and universally closed. By [Sta14, 081T], there is a $V \setminus U$-admissible blow-up of V that dominates $\tilde{V}$. Therefore $X \to S$ is proper. ☐

4.7 Embeddings are proper

Recall (def. 3.6) that an open immersion followed by an embedding is called an *immersion*. Every immersion is separated. Since open immersions can be understood, by definition, at the level of point-set topology, so too can the difference between immersions and embeddings. In particular:

4.40 Lemma. *A surjective immersion is an embedding.*

4.41 Lemma. *Let* $X \hookrightarrow S$ *be an immersion,* $U \hookrightarrow V$ *an affine open immersion of* S*-schemes. Then* $\mathrm{Hom}_S(\mathrm{cl}(U/V), X) \tilde{\to} \mathrm{Hom}_S(\mathrm{Sur}_{U/V}, X)$*.*

Proof. Without loss of generality, assume $V = \mathrm{cl}(U/V)$. Let $\tilde{V}$ be an overconvergent neighbourhood of U/V, $\tilde{V} \to X$ an extension.

The base change $X \times_S V \to V$ is an immersion containing U. It is also surjective, since $\tilde{V} \to V$ is surjective. Therefore it is an isomorphism by lemma 4.40. ☐

It follows, more or less tautologically:

4.42 Proposition. *An immersion is proper if and only if it is an embedding.*

A formal immersion is $f\mathbb{P}$*-proper if and only if it is a formal embedding.*

Proof. Let $X \hookrightarrow S$ be a formal embedding, and let U/V be an elementary extension problem. Then in particular, X/S has affine diagonal, and so $U \to X$ is affine. Since taking the formally embedded image is order-preserving,

$$\mathrm{cl}(V/S) = \mathrm{cl}(U/S) = \mathrm{cl}(\mathrm{cl}(U/X)/S) \subseteq X.$$

Thus $V \to X$ and X is $f\mathbb{P}$-proper.

Conversely, let $X \hookrightarrow S$ be a formally proper formal immersion, and let $U \to S$ be an affine formal immersion factoring through X. Let $V = \mathrm{cl}(U/S)$ be the formally embedded closure; then U is open in V. By the definition of propriety, there is a projective modification of V that factors through X. Since U/V is dense, such a modification is necessarily surjective. Therefore $V \subseteq X$.

The first statement follows immediately from the second and the definitions. ☐

We will return to a discussion on this theme in [Mac15, §4.5].

4.8 Alternative characterisations

In this section, we gather a few alternative characterisations of separated, proper, and overconvergent morphisms couched in more traditional terms.

4.8.1 Diagonal criterion for separation

4.43 Corollary (of prop. 4.42)**.** *Let $X \to S$ be a morphism of formal schemes. The following are equivalent:*

i) *X/S is separated;*

ii) *the diagonal $X \to X \times_S X$ is an embedding.*

Proof. Indeed, the proof of [Sta14, 01KJ] runs without modification in the $\mathbb{F}_1$ setting to establish that the diagonal is always an immersion. □

4.44 Aside. Of course, we cannot ask that the diagonal be closed: this fails even for the affine line over $\mathbb{F}_1$, cf. [Dur07, §6.5.20].

The diagonal of a separated morphism can also fail to be affine, as the following pathological example shows: Let $V = \mathrm{Spec}(\mathbb{F}_1 \times \mathbb{F}_1)$, $U \subset V$ the complement of the closed point. Then $U \hookrightarrow V$ is projective, but not affine. Thus the scheme obtained by glueing two copies of V along U is separated with non-affine diagonal.

4.8.2 Chow criterion for propriety

The Chow lemma depends on the fact that $\mathbf{FSch}_S^{\mathrm{ltf}}$ has a site consisting of open subobjects of objects $\mathbb{P}$ over the base.

4.45 Theorem (Strong Chow lemma)**.** *A finite type morphism $X \to S$ is proper if and only if it is quasi-separated and, for any $U \subseteq X$ quasi-projective over S, there locally on S exists an extension*

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "\exists\mathbb{P}"'] \arrow[ddr, bend left, "\mathbb{P}"] & \\
U \arrow[ur, bend left, "\exists"] \arrow[r, hook, "\forall\circ"] \arrow[drr, bend right, "quasi\text{-}\mathbb{P}"'] & X \arrow[dr] & \\
 & & S
\end{tikzcd}$$

of the inclusion of U in X to a $\mathbb{P}$ morphism $\tilde{X} \to X$ such that $\tilde{X}$ is $\mathbb{P}$ over S, and moreover this property persists after any base change.

Proof. Indeed, this Chow property states that for any quasi-projective U and quasi-compact open immersion $U \hookrightarrow V$, the canonical solution $X \times V \to V$ can be dominated by a $\mathbb{P}$ morphism with a section over U and is therefore an overconvergent neighbourhood. □

4.8.3 Local criterion for overconvergence

Overconvergence is a bit more difficult to characterise in classical terms, since there is little literature on the subject.

4.46 Proposition (Overconvergence is local propriety)**.** *Let $X \to S$ be a quasi-separated morphism of formal schemes. The following are equivalent:*

i) f is overconvergent;

ii) every proper formal X-scheme qcqs over S is proper over S;

iii) every formally embedded formal subscheme of X qcqs over S is proper over S.

Proof. The implications *i)⇒ii)⇒iii)* being clear, suppose *iii)*, and let U/V be an elementary extension problem. Since X is quasi-separated over S, $U \to X$ is quasi-compact. There is therefore a formally embedded subscheme Z of X qcqs over S that contains the image of U. By hypothesis, it is proper and so $\mathrm{Sur}_{U/V} \to Z \hookrightarrow X$. □

A morphism between locally integral over Noetherian formal schemes is always quasi-separated. Hence, this criterion for overconvergence applies in most cases one encounters in practice in formal geometry.

4.8.4 Valuative criteria

Here, we will establish a version of the *valuative criteria* of [Gro60, **II**.7] for Noetherian formal $\mathbb{F}_1$-schemes.

4.47 Lemma. *Let V be an affine $\mathbb{F}_1$-scheme, $U = \operatorname{Spec} A \hookrightarrow V$ a dense affine open immersion. Let $U_{\mathbb{F}_1} = \operatorname{Spec} A/(A^\times = 1)$, $V_{\mathbb{F}_1}$ the localisation. Then*

$$\begin{tikzcd}
U_{\mathbb{F}_1} \arrow[r] \arrow[d] & V_{\mathbb{F}_1} \arrow[d] \\
U \arrow[r] & V
\end{tikzcd}$$

is a pushout in the category of algebraic spaces.

Proof. Since the vertical maps are homeomorphisms, it will be enough to treat the affine case; that is, we must check that the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A[f^{-1}] \arrow[d] \\
A/A^\times \arrow[r] & A/A^\times[f^{-1}]
\end{tikzcd}$$

is Cartesian. This follows from the fact that the inclusion $A \subseteq A[f^{-1}]$ is preserved by the group action of $A^\times$. □

4.48 Theorem (Valuative criterion)**.** *Let S be a locally Noetherian formal $\mathbb{F}_1$-scheme, $f : X \to S$ locally of finite type and paracompact. Then f is overconvergent if and only if every commuting square*

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{F}_1} \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{F}_1} \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

admits a unique lift $\mathbb{A}^1_{\mathbb{F}_1} \to X$.

If S is more generally locally integral / Noetherian, then the same holds with $\mathbb{A}^1_{\mathbb{F}_1}$ replaced with $\operatorname{Spec}\mathbb{F}_1[z^{\mathbb{Q}}]$.

Proof. Indeed, by corollary 5.10, we may assume that U and V are quasi-integral and, replacing X with the closure of the image of U, that X is quasi-integral and Noetherian. The result then follows from proposition 6.11. □

I had initially hoped to prove this result directly by an algorithmic construction based on proposition 4.29, but my attempts were ultimately confounded, and so the proof given rests on the classification of $\mathbb{F}_1$-schemes and criterion for propriety found in §6. In particular, the methods manifestly do not apply over $\mathbb{Z}$. We do not use this criterion except in the proof of theorem 4.53, which is independent of the rest of the paper.

4.49 Aside. This does not imply that overconvergence can be detected by *non-boundary* morphisms from $\mathbb{A}^1_{\mathbb{F}_1}$. Indeed, the fan in $\mathbb{Q}^2$ whose non-zero cones are every rational ray provides a counterexample to that statement.

The valuative criterion also fails without the paracompactness assumption: indeed, the construction $U^{\text{él}}$ (def. 4.23) applied to the affine plane provides a counterexample.

4.8.5 Overconvergence via images

We have already seen (prop. 4.39) that proper morphisms over $\mathbb{Z}$ can be characterised, as in [Gro60], by the behaviour of closed subsets under images. Over $\mathbb{F}_1$, it is of course not true that proper morphisms are universally closed, and since embeddings are not characterised by their underlying sets, one must be a bit more careful when taking the image.

In this section I present a possible analogue to the approach of Grothendieck.

4.50 Definition. Write $\hat{\mathfrak{Z}}(X/S)$ for the poset of formally embedded subschemes of X qcqs over S. Let us call a morphism $f : X \to S$ *grounded* if for every $Z \in \hat{\mathfrak{Z}}(X/S)$,

$$Z \to \mathrm{cl}(Z/S)$$

is surjective on points. It is *universally grounded* if it is grounded after any base change.

The intuition behind this definition is that if it were possible to define the formally *immersed* image of morphisms, then by lemma 4.40, the difference between this and the embedded image would be detected by the underlying set. Indeed, one can show that grounding implies that the square

$$\begin{tikzcd}
\hat{\mathfrak{Z}}(X/S) \arrow[r] \arrow[d, "\mathrm{cl}(-/S)"'] & \mathrm{Imm}(X/S) \arrow[d, "f_!"] \\
\mathrm{Pro}\hat{\mathfrak{Z}}(S) \arrow[r] & \mathrm{ProImm}(S)
\end{tikzcd}$$

commutes, where $\mathrm{Imm}(X/S)$ denotes the poset of formally immersed subschemes of X qcqs over S, and $f_!$ the left pro-adjoint to pullback.

With this definition, it is easy to establish an analogue of [Gro60, **II**.5.2.3.*ii)*] - which would be useful to have for our definition of propriety via extension problems:

4.51 Lemma. *Let $X \to Y \to S$ be a pair of morphisms such that*

- *i) $X \to S$ is universally grounded;*
- *ii) $X \to Y$ is* strongly surjective*, meaning that for any embedded subscheme $Z \hookrightarrow Y$,*

$$Z = \mathrm{cl}(Z \times_Y X/Y).$$

Then $Y \to S$ is universally grounded.

4.52 Example. Note that surjectivity of X/Y is clearly not enough for this lemma to run in the $\mathbb{F}_1$ case, as the example $\{1\} \to \mathbb{A}^1_{\mathbb{F}_1} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{F}_1}$ demonstrates.

4.53 Theorem. *Let S be locally integral / Noetherian, $f : X \to S$ separated, of finite type, and universally grounded. Then f is proper.*

Proof. We apply the valuative criterion. For ease of notation, we treat the Noetherian case, though the general case follows from the same argument. Consider a commuting square

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{F}_1} \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{F}_1} \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

and replace X resp. S with the embedded closures of $\mathbb{A}^1_{\mathbb{F}_1} \setminus \{0\}$ resp. $\mathbb{A}^1_{\mathbb{F}_1}$; this is possible because every open subset of $\mathbb{A}^1_{\mathbb{F}_1}$ is affine, and so the morphisms to X and S are also affine. By hypothesis, $X \to S$ is now surjective.

We therefore obtain a dual square

$$\begin{tikzcd}
\mathscr{O}(S) \arrow[r] \arrow[d] & \mathscr{O}(X) \arrow[d] \\
\mathbb{F}_1[t] \arrow[r] & \mathbb{F}_1[t^{\pm 1}]
\end{tikzcd}$$

in which all arrows are injective. Thus $\mathscr{O}(X) \setminus 0$ is a submonoid of $\mathbb{Z}$ and $\mathscr{O}(S) \setminus 0$ of $\mathbb{N}$. By surjectivity of f, $\mathscr{O}(X) \setminus 0$ must in fact be contained in $\mathbb{N}$. Therefore $\mathbb{A}^1_{\mathbb{F}_1} \to X$. □

5 Integrality and normalisation

In this section we discuss some absolute properties of $\mathbb{F}_1$-scheme that bring us towards the realm of toric geometry:

- quasi-integral;
- integral;

- normal.

The *quasi-integral* condition is an intermediate notion that does not appear for ordinary rings; it exists because a monoid can fail to be cancellative without having zero-divisors. This is another manifestation of the fact that embeddings can fail to be closed. The geometric provenance of such monoids appears to be quite mysterious.

On the other hand, with *non-boundary morphisms* (morphisms with no kernel) one can define an interesting subcategory of quasi-integral $\mathbb{F}_1$-schemes that enjoys stability under fibre products and localisations; no analogous construction can work over $\mathbb{Z}$, or with the more restrictive hypotheses of integrality or normality.

5.1 Quasi-integral

5.1 Definition. An $\mathbb{F}_1$-algebra A is said to be *quasi-integral* if the zero ideal is prime, or equivalently, if $A \setminus 0$ is a submonoid. A homomorphism of $\mathbb{F}_1$-algebras is said to be *non-boundary* if its kernel is zero.

An $\mathbb{F}_1$-scheme is quasi-integral if the stalks of its structure sheaf are quasi-integral $\mathbb{F}_1$-algebras. A morphism $f : X \to Y$ of $\mathbb{F}_1$-schemes is non-boundary if $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ has zero kernel. The category of quasi-integral schemes and non-boundary homomorphisms is denoted $\mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$.

The subcategory $\text{Alg}_{\mathbb{F}_1}^{\text{qi/nb}}$ of $\text{Alg}_{\mathbb{F}_1}$ consisting of quasi-integral algebras and non-boundary morphisms is exactly the image of the category of ordinary monoids under adjoining zero: the restriction of this functor

$$\mathbb{F}_1[z^-] : \mathbf{Mon} \to \text{Alg}_{\mathbb{F}_1}^{\text{qi/nb}}$$

is an equivalence with inverse $A \mapsto A \setminus 0$. It follows that any colimit of quasi-integral $\mathbb{F}_1$-algebras along non-boundary homomorphisms is quasi-integral.

In particular, the inclusion of opposite categories is left exact, and so generates a subtopos $\text{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$ under colimits. The inclusion is the pullback functor for an essential geometric morphism

$$\text{Sh}\mathbf{Sch}_{\mathbb{F}_1} \to \text{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$$

whose pullback preserves open immersions. The topos on the right is nothing more than the presheaf topos on the opposite category to $\text{Alg}_{\mathbb{F}_1}^{\text{qi/nb}}$. The associated embedding of the category of locally representable objects into $\mathbf{Sch}_{\mathbb{F}_1}$ identifies it with $\mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$, since being quasi-integral or non-boundary is a local property.

5.2 Aside. Note that an affine scheme is quasi-integral if and only if it is the spectrum of a quasi-integral $\mathbb{F}_1$-algebra, *except for the empty set*, which is locally representable, but not representable, in $\text{Alg}_{\mathbb{F}_1}^{\text{qi/nb}}$.

5.3 Lemma. *A scheme is quasi-integral if and only if every open subset is non-boundary.*

Proof. It suffices to check the affine case. The kernel of a localisation $A \to A[f^{-1}]$ is the annihilator of f. Quasi-integrality of A means that this is zero whenever f is non-zero. If $f = 0$, then this is dual to the inclusion of $\emptyset \in \mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$. □

In other words, the small site of a quasi-integral scheme does not depend on whether we consider it as an object of $\text{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\text{qi/nb}}$ or of $\text{Sh}\mathbf{Sch}_{\mathbb{F}_1}$.

Detecting overconvergence

5.4 Lemma. *i) The total space of a blow-up of a quasi-integral scheme is quasi-integral, and the projection is non-boundary.*

ii) Let U/V be an open immersion of quasi-integral schemes. The relative normalisation $\nu_{V\setminus U}V$ is quasi-integral, and the projection is non-boundary.

It follows that for $U \hookrightarrow V$ in $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$, the overconvergent germ $\mathrm{Sur}_{U/V}$, computed in $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$, is also a pro-object of $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$.

5.5 Proposition. *The pullback* $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}} \hookrightarrow \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$ *preserves overconvergence.*

Proof. Let U/V be an extension problem for a non-boundary morphism $X \to S$ of quasi-integral schemes. We will produce an extension problem U_0/V_0 in $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$ factoring U/V and such that $U \cong U_0 \times_{V_0} V$. This will get us criterion *iv)* of lemma 4.3.

Suppose that V is affine and that $U \hookrightarrow V$ is affine and dense, and replace X with a quasi-integral affine scheme through which the morphism from U factors. Then we are in the situation of a commuting square

$$\begin{tikzcd}
\mathcal{O}_S(V)[f^{-1}] & \mathcal{O}_S(V) \arrow[l, hook'] \\
\mathcal{O}_S(X) \arrow[u] & \mathcal{O}_S \arrow[l] \arrow[u]
\end{tikzcd}$$

with the $\mathbb{F}_1$-algebras in the bottom row integral and $f \in \mathcal{O}_S(V)$ a cancellable element.

Let us define $\mathcal{O}_S(U_0)$ to be the subalgebra of $\mathcal{O}_S(U)$ generated by $\mathcal{O}_S(X)$ and $f^{\pm 1}$, and $\mathcal{O}_S(V_0) = \mathcal{O}_S(U_0) \cap \mathcal{O}_S(V)$. These subalgebras are quasi-integral because f is a non-zero-divisor and $\mathcal{O}_S(X)$ is quasi-integral. The homomorphisms between them are injective, and so the dual square

$$\begin{tikzcd}
U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

now lives in $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$. By lemma 4.20, $U_0 \hookrightarrow V_0$ is an open immersion. □

5.2 Irreducible components

If X is any $\mathbb{F}_1$-scheme and $I \trianglelefteq \mathcal{O}_X$ an ideal sheaf, then the closed subscheme cut out by I is quasi-integral if and only if I is prime. This defines a one-to-one, inclusion-reversing correspondence

$$\{\text{quasi-integral closed subschemes of } X\} \quad \leftrightarrow \quad \{\text{primes of } \mathcal{O}_X\}.$$

Now suppose the underlying topological space of X is Noetherian and let $X = \bigcup_{i=1}^{k} X_i$ be its decomposition into irreducible components. Each of these components is the closure of a unique minimal prime $\mathfrak{p}_i \trianglelefteq \mathcal{O}_X$. Equipped with their reduced scheme structure they are quasi-integral closed subschemes. We will always consider the irreducible components of X - when they exist - to carry this scheme structure.

Categorical decomposition into irreducible components Suppose that $\{T_i \trianglelefteq A\}_{i\in J}$ is a finite family of ideals in an algebra A (over $\mathbb{F}_1$ or $\mathbb{Z}$). Let's write

$$T_I := \sum_{i\in I} T_I$$

for a subset $I \subset J$, and $T = \bigcap_{i\in J} T_j$. The family A/T_I is then filtered.

Over $\mathbb{Z}$, it is possible to show that

$$A/T \xrightarrow{\sim} \lim_I A/T_I$$

in the category of rings. In other words, a finite union of closed subschemes of a fixed scheme is actually a *colimit* in $\mathbf{Sch}_{\mathbb{Z}}$. In particular, a scheme over $\mathbb{Z}$ whose underyling topological space is Noetherian has a *categorical* decomposition into irreducible components.

Unfortunately, for schemes relative to the category of monoids this cannot be true.

5.6 Example (More stupid limits of monoids). Let X be the spectrum of $\mathbb{F}_1[x,y]/xy$. The irreducible components of this thing are the axes, intersecting in the origin, so we are led to consider the square

$$\begin{tikzcd}
\mathbb{F}_1[x,y]/xy \arrow[r, "x=0"] \arrow[d, "y=0"'] & \mathbb{F}_1[y] \arrow[d] \\
\mathbb{F}_1[x] \arrow[r] & \mathbb{F}_1
\end{tikzcd}$$

Unfortunately, this square is *not* Cartesian - in fact, the pullback monoid has infinitely many transcendents.

Taking the limit instead in the category of monads improves matters slightly: we get an additional relation $w = x + y$. The prime spectrum of this monad has the same underlying space as X, although there is still no morphism to it from X. There is therefore some hope that certain schemes relative to the category of monads (or blueprints, or sesquiads) will admit a categorical decomposition into irreducible components.

In any case, we are able to get a weaker result in the present framework which is sufficient for understanding overconvergence:

5.7 Proposition. *Let $U \hookrightarrow V$ be an affine, dense open immersion. Let $\{V_i \subseteq V\}_{i\in J}$ be a finite family of closed subschemes with defining ideals $T_i \trianglelefteq \mathscr{O}_V$, and suppose that $U \cap V_i$ remains scheme-theoretically dense in V_i for each i.*

Let $\tilde{V}$ be the subscheme cut out by $\bigcap_{i\in J} T_i$. Then

$$\begin{tikzcd}
\mathrm{colim}_I(U \cap V_I) \arrow[r] \arrow[d] & \mathrm{colim}_I V_I \arrow[d] \\
U \cap \tilde{V} \arrow[r] & \tilde{V}
\end{tikzcd}$$

is a pushout in the category of algebraic spaces.

Proof. Let X be an algebraic space, and take a morphism to X from the pushout. After replacing V with an affine cover, we will need to produce an affine open subset of X through which $U \cap \tilde{V}$ and all the V_i factor. We may then reduce to the affine case. That the resulting diagram is a pushout in the category of affine schemes is the content of lemma 5.8.

Since affine $\mathbb{F}_1$-schemes are local, V has a unique closed point, contained in each V_i. It will be enough to show that the closed points of all the V_i determine the same point of X. But this follows from the fact that the strata V_I form a cofiltered poset. □

5.8 Lemma. *Let $A \hookrightarrow A[f^{-1}]$ be a localisation at a cancellable element f. Suppose that $f \not\in \bigvee_{i\in J} T_i$. Then*

$$\begin{tikzcd} A/\bigcap_{i\in J} T_i \arrow[r] \arrow[d] & A[f^{-1}]/\bigcap_{i\in J} T_i \arrow[d] \\ \prod_i A/T_i \arrow[r] & \prod_i A[f^{-1}]/T_i \end{tikzcd}$$

is a pullback.

Proof. The assumptions ensure that all the arrows are injective, so we are investigating an intersection of sets. Let $g \in A[f^{-1}]/\bigcap_i T_i$ be non-zero. Then g is also non-zero in some $A[f^{-1}]/T_i$. If its image there is contained in A/T_i, then via the set-theoretic section

$$(A/T_i)\setminus 0 \to (A/\cap_i T_i)\setminus T_i$$

it is also contained in $A/\bigcap_{i\in J} T_i$. □

5.9 Corollary. *Let X be a reduced integral/Noetherian $\mathbb{F}_1$-scheme, $U \subseteq X$ a dense open subset. Then X is the colimit in the category of schemes of U and its quasi-integral components.*

Proof. By the proposition, it is enough to show that the intersection of the minimal primes of $\mathscr{O}_X$ is its nilradical, which follows from the usual argument in commutative algebra. □

5.10 Corollary. *Let S be locally integral-over-Noetherian, $X \to S$ locally of finite type. Then X/S is overconvergent if and only if every extension problem with quasi-integral test spaces has a unique solution.*

Proof. Let U/V be an elementary extension problem 4.13. By propositions 4.33 and 4.30, we may assume that V is a reduced scheme of finite type over S. Then by lemma 2.4 the underlying topological spaces of S, U, V, and X are (locally in the latter case) Noetherian.

Let us write $V = \bigcup_i V_i$ as a union of irreducible quasi-integral closed subschemes.Write $U_i := U \times_V V_i$, and let $\mathrm{Sur}_{\coprod_i (U_i/V_i)} \to X$. By theorem 3.26, we may assume that the morphism is defined on a finite type blow-up $\coprod_i \tilde{V}_i \to \coprod_i V_i$ along some ideals $T_i \trianglelefteq \mathscr{O}_{V_i}$.

Considering these as ideals on V via the closed embeddings $V_i \hookrightarrow V$, let $T = \prod_i T_i$ and let $\tilde{V}$ be the ($V \setminus U$-admissible) blow-up. By lemma 5.7, we obtain a unique morphism to X from the union of the closed subschemes $\tilde{V}_i \times_V \tilde{V}$ of $\tilde{V}$, extending the given $\tilde{V}_i \to X$. Since the union is finite, the inclusion of the resulting subscheme is a closed embedding, hence in particular projective. This therefore determines a unique map $\mathrm{Sur}_{U/V} \to X$ descending from $\mathrm{Sur}_{\coprod_i (U_i/V_i)} \to X$. □

5.3 Integrality and normalisation

Let A be quasi-integral. The *field of fractions* of A is the localisation

$$K_A := \mathbb{F}_1[z^{(A\setminus 0)\otimes\mathbb{Z}}].$$

The fraction fields glue together to yield a sheaf K of $\mathbb{F}_1$-fields on $\mathbf{Sch}^{\mathrm{aff/qi/nb}}$.

If X is a connected, quasi-integral $\mathbb{F}_1$-scheme, then non-empty open subsets are topologically dense, and so the restriction K_X of K to (the small site of) X is constant. In this case, we will confuse K_X with its algebra of sections, which is called the *function field* of X. Its spectrum is the *generic point* of X. The inclusion of the generic point is an affine morphism.

This allows us to define absolute versions of the notions introduced for pairs in §3.2.

5.11 Definition (Integrality)**.** A quasi-integral $\mathbb{F}_1$-algebra A is *integral* if every non-zero principal divisor is Cartier; that is, if $A\setminus 0$ is a cancellative monoid.

An integral $\mathbb{F}_1$-algebra A is *normal* if $(K_A;A)$ is relatively normal (def. 3.13); that is, if $A\setminus 0$ is saturated in $(A\setminus 0)\otimes\mathbb{Z}$.

A quasi-integral $\mathbb{F}_1$-scheme X is said to be *integral*, resp. *normal*, if the stalks of $\mathscr{O}_X$ are integral, resp. normal, $\mathbb{F}_1$-algebras.

One associates to any quasi-integral $\mathbb{F}_1$-scheme an *underlying integral subscheme*, which for connected X is calculated as the affine embedding

$$X^{\mathrm{i}}\cong\mathrm{cl}(\operatorname{Spec}K_X/X)\hookrightarrow X.$$

In particular, a quasi-integral scheme is integral if and only if every inhabited open subset is scheme-theoretically dense.

Similarly, an integral scheme may be replaced with its *normalisation* $\nu X\to X$, which on connected components is the same as the relative normalisation of the pair $(\operatorname{Spec}K_X,X)$. Note that the normalisation and the inclusion of the underlying integral subscheme are *homeomorphisms* in the $\mathbb{F}_1$-setting. They are integral/projective.

These constructions yield right adjoints to the inclusions

$$\mathbf{Sch}^{\mathrm{n/nb}}_{\mathbb{F}_1}\hookrightarrow\mathbf{Sch}^{\mathrm{i/nb}}_{\mathbb{F}_1}\hookrightarrow\mathbf{Sch}^{\mathrm{qi/nb}}_{\mathbb{F}_1}$$

of the full subcategories of $\mathbf{Sch}^{\mathrm{qi/nb}}_{\mathbb{F}_1}$ whose objects are normal, respectively integral schemes.

5.12 Aside. The categories of integral and normal $\mathbb{F}_1$-schemes are not closed in $\mathbf{Sch}^{\mathrm{qi/nb}}_{\mathbb{F}_1}$ under fibre products, as the co-Cartesian square

$$\begin{tikzcd}
\mathbb{F}_1[x,y] \arrow[r, "y/x=w"] \arrow[d, "y=x"'] & \mathbb{F}_1[x,w] \arrow[d] \\
\mathbb{F}_1[x] \arrow[r] & \mathbb{F}_1[x,w]/(xw=w)
\end{tikzcd}$$

easily demonstrates. (This example is an affine patch of the pullback of the diagonal of $\mathbb{A}^2$ along the blow-up at 0).

5.13 Lemma. *Passage to the underlying integral scheme* $\mathbf{Sch}^{\mathrm{qi/nb}}_{\mathbb{F}_1}\to\mathbf{Sch}^{\mathrm{i/nb}}_{\mathbb{F}_1}$ *detects and preserves overconvergence.*

5.14 Lemma. *Normalisation* $\mathbf{Sch}^{\mathrm{i/nb}}_{\mathbb{F}_1}\to\mathbf{Sch}^{\mathrm{n/nb}}_{\mathbb{F}_1}$ *detects and preserves i/*$\mathbb{P}$*-overconvergence.*

Proof. We prove both lemmas at once. The preservation statements follow from the fact that normalisation is of class $i/\mathbb{P}$, and so the overconvergent germ $\mathrm{Sur}_{U/V}$ does not depend on in which category it is computed.

Now let U/V be an affine, dense extension problem, and let νU, resp νV, be the normalisation of U, resp. V. Define $\tilde{V}$ via the pullback square

$$\begin{tikzcd}
\mathcal{O}(\tilde{V}) \arrow[r] \arrow[d] & \mathcal{O}(U) \arrow[d] \\
\mathcal{O}(\nu V) \arrow[r] & \mathcal{O}(\nu U).
\end{tikzcd}$$

The dual square is a pushout in $\mathbf{Sch}_{\mathbb{F}_1}$ because $\nu V \to \tilde{V}$ is a homeomorphism. Moreover, $\tilde{V} \to V$ is integral/projective. This shows criterion *iv)* of lemma 4.11. □

6 From schemes to fans

The starting point of the classification theorem is the observation, first codified, to the best of my knowledge, in [Dei08], that integral $\mathbb{F}_1$-schemes are essentially the same as toric varieties: they can be packaged in terms of a *fan* in a rational vector space.

Most of the statements in this section are well-known basic properties of toric geometry. I restate them here with an eye towards generalisation to the formal and rigid analytic cases. In this and the next section, we state relative properties of morphisms as an absolute property of an $\mathbb{F}_1$-scheme X if it holds for the structural morphism $X \to \mathrm{Spec}\,\mathbb{F}_1$.

The main thrust will be to explain the following statement:

6.1 Theorem. *The construction*

$$X \mapsto (K_X^\times, \Sigma_X)$$

sets up an equivalence between the category of normal, connected, separated $\mathbb{F}_1$-schemes with enough jets, with non-boundary morphisms, and the category of pairs consisting of an Abelian group and a fan in its $\mathbb{Q}$-dual. Moreover, we have the following equivalences:

- *i) X is quasi-compact if and only if Σ_X is finite;*
- *ii) X is locally integral / Noetherian if and only if the cones of Σ_X are rational polyhedral;*
- *iii) X is locally Noetherian if and only if in addition to ii), Σ_X is spanned by its $\mathbb{Z}$-points;*
- *iv) X is proper if and only if in addition to iii), $\Sigma_X(\mathbb{R}) = N(\mathbb{R})$.*

The integral closed subschemes of X are in natural, inclusion-reversing correspondence with the cones of Σ_X, with the fan data of a closed subscheme with cone σ given by $(K_X^\times/\sigma, \Sigma_X/\sigma)$.

The notation will be defined shortly, though its meaning should be clear to students of toric geometry (see [Ful93]). A version of this, missing the separation hypothesis, appeared as [Dei08, thm. 4.1].

6.1 Diagonalisable group action

The category of monoids, and hence $\mathrm{Alg}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$, is semiadditive, that is, finite products are finite coproducts. It follows that any quasi-integral $\mathbb{F}_1$-algebra A is naturally a *bialgebra*, with comultiplication given by the diagonal

$$A \to A \otimes_{\mathbb{F}_1} A, \quad f \mapsto f \otimes f$$

and counit by the homomorphism $A \to \mathbb{F}_1$ that sends all non-zero elements to 1. Non-boundary homomorphisms are automatically bialgebra homomorphisms. If $A = K_A$ is an $\mathbb{F}_1$-field - so $A \setminus 0$ is an Abelian group - it is even a Hopf algebra.

Thus any affine, quasi-integral $\mathbb{F}_1$-scheme X carries the structure of a *monoid scheme* over $\mathbb{F}_1$, with unit the unique $\mathbb{F}_1$-point sitting over the generic point of X, and this structure is automatically compatible with non-boundary morphisms. In particular,

$$\mathbb{G}_X := \mathrm{Hom}(K_X^\times, \mathbb{G}_m) = \mathrm{Spec} K_X$$

carries the structure of a *group* $\mathbb{F}_1$*-scheme*. An $\mathbb{F}_1$-field is determined by its *character group*

$$K_A^\times = \mathrm{Hom}(\mathrm{Spec} K_A, \mathbb{G}_m) = K_A \setminus 0$$

where of course write $\mathbb{G}_m$ for the group scheme $\mathrm{Spec}\mathbb{F}_1[t^{\pm 1}]$.

More generally, $\mathbb{G}_X$ is defined for any connected, quasi-integral $\mathbb{F}_1$-scheme X and acts naturally, so for any non-boundary homomorphism $X \to Y$ the square

$$\begin{tikzcd}
\mathbb{G}_X \times X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{G}_Y \times Y \arrow[r] & Y
\end{tikzcd}$$

commutes. We refer to $K_X^\times$ also as the *character group of* X.

We will also see that for irreducible closed subscheme $Z \hookrightarrow X$, there is a natural quotient homomorphism $\mathbb{G}_X \to \mathbb{G}_Z$ such that the diagram

$$\begin{tikzcd}
\mathbb{G}_X \times Z \arrow[r] \arrow[d] & \mathbb{G}_Z \times Z \arrow[r] & Z \arrow[d] \\
\mathbb{G}_X \times X \arrow[rr] & & X
\end{tikzcd}$$

commutes.

Of course, this action is preserved - and becomes visible at the level of underlying topological spaces - after base change to $\mathbb{Z}$. In other words, any scheme with a quasi-integral model over $\mathbb{F}_1$ inherits an action by an equivariantly embedded diagonalisable group. The classification theorem in this section can therefore be thought of as a classification of ordinary schemes with this extra structure.

In particular, if X is a normal, separated, and of finite type, and $K_X^\times$ is torsion free, then its base change is a *toric variety* in the sense of [Ful93], and any non-boundary homomorphism yields a *toric morphism* of the base changes.

6.2 Embedded cones

Suppose again that $X = \operatorname{Spec} A$ is affine. Then $A \setminus 0$ is a generating cone in the character group $K_A^\times$. Its *polar*

$$\sigma_A = \{v : K_A^\times \to \mathbb{Q} | v(A \setminus 0) \leq 0\}$$

is a strongly convex cone in the rational vector space $N_A(\mathbb{Q}) := \operatorname{Hom}(K_A^\times, \mathbb{Q})$. This sets up a functor

$$\Sigma : \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}} \to \mathbf{Cone}^N, \quad \operatorname{Spec} A \mapsto (K_A^\times, \sigma_A)$$

from the category of affine quasi-integral $\mathbb{F}_1$-schemes with non-boundary morphisms to the category $\mathbf{Cone}^N$ of pairs $(\sigma, K^\times)$ consisting of an Abelian group $K^\times$ and a strongly convex, reflexive cone σ in its rational dual $N(\mathbb{Q})$.

More generally, we treat N as a functor of Abelian groups, writing $N(H) = \operatorname{Hom}(K^\times, H)$ for a group H. When H is totally ordered, the points of $N(H)$ can be thought of as *valuations* on K. Each non-zero rational function $f \in K^\times$ induces a linear function $N(H) \to H$, denoted $\log f$, and this correspondence is injective as long as $K^\times$ is torsion-free.

In toric parlance, $N_A(\mathbb{Z})$ is the group of cocharacters, or 1-parameter subgroups, of $\mathbb{G}_A$.

Open subsets The map of cones associated to a localisation $A \to A[f^{-1}]$ is the inclusion of the face of σ_A where $\log f = 0$.

Let us define a *face* of an object $(\sigma, K^\times)$ of $\mathbf{Cone}^N$ to be a pair $(\tau, K^\times)$, where τ is a subcone of σ defined by a collection of functions $f \in K^\times$ such that $\log f \leq 0$ on σ. Such a sub-cone is automatically strongly convex and reflexive. We call a face *principal* if it can be cut out by a single function f. Of course, in the case $\sigma = \sigma_A$, then any element of A cuts out a principal face of σ_A.

Then Σ induces a bijection between the set of open immersions into $\operatorname{Spec} A$ and the set of principal faces of σ_A.

Closed subsets Let $\mathfrak{p} \trianglelefteq A$ be a prime ideal. Then $A/\mathfrak{p}$ is in canonical bijection with $A \setminus \mathfrak{p} \subseteq A \setminus 0$, and so $K_{A/\mathfrak{p}}^\times \subseteq K_A^\times$. We obtain surjective maps

$$N_A \twoheadrightarrow N_{A/\mathfrak{p}}, \quad \sigma_A \twoheadrightarrow \sigma_{A/\mathfrak{p}}$$

with kernel the subspace of N_A of valuations *centred at* $\mathfrak{p}$, that is, that restrict to zero on $A \setminus \mathfrak{p}$. The intersection of this kernel with σ_A is the intersection of the faces cut out by the functions $\log f$ for $f \in \mathfrak{p}$.

This correspondence is natural in $\mathfrak{p}$, and hence puts the set of irreducible closed subsets of $\operatorname{Spec} A$ in inclusion-reversing correspondence with the set of linear quotients of $(K_A^\times, \sigma_A)$ by faces of σ_A.

Since any morphism can be written as a non-boundary morphism followed by a closed embedding, embedded cones can be used to describe all morphisms between affine quasi-integral $\mathbb{F}_1$-schemes.

Points Let H be an Abelian group. In order to make sense of the set $\sigma(H) \subseteq N(H)$ of H-rational points of the cone σ, we need a partial order on H. Thus σ can be thought of as a functor of partially ordered groups (in the sequel, *pogroups*). For concreteness, the reader may like to focus on the case that H is totally ordered and Archimedean, i.e. of rank one.

The free $\mathbb{F}_1$-algebra on H, when equipped with the t-adic topology, is called an $\mathbb{F}_1$-*valuation field* and denoted $\mathbb{F}_1(\!(t^{-H})\!)$. Its *ring of integers*, comprised of negative elements of H, or positive powers of t, is denoted $\mathbb{F}_1[\![t^{-H}]\!]$. The underlying discrete $\mathbb{F}_1$-algebra is what we would have called $\mathbb{F}_1[z^{H_{\leq 0}}]$ in the notation of §2.

One usually thinks of the spectrum Δ of $\mathbb{F}_1[\![t]\!] = \mathbb{F}_1[\![t^{\mathbb{Z}}]\!]$ as a *formal disc*; we will generalise this to arbitrary H and write

$$\Delta_H := \operatorname{Spec}\mathbb{F}_1[\![t^{-H}]\!]$$

for the formal disc and punctured disc with exponent group H. Morphisms from Δ (resp. Δ_H) into a formal scheme are called *jets* (resp. *H-jets*). While we are considering only schemes, the topology of $\mathbb{F}_1[\![t^{-H}]\!]$ plays no rôle.

Geometrically, the set $\sigma_A(H)$ of valuations from K_A into H, non-positive on A, correspond to non-boundary H-jets $\Delta_H \to \operatorname{Spec}A$. The point sets are a left exact functor.

Topological realisation The vector space $N(\mathbb{R})$ carries a natural weak topology coming from the order topology on $\mathbb{R}$, and this is inherited by $\sigma(\mathbb{R})$, making it a contractible $\mathbb{R}^\times_{\geq 0}$-space. It also inherits an H-affine structure - in particular, a subset of H-rational points - for any additive subgroup $H \subseteq \mathbb{R}$.

The topological realisation is a left exact functor, and $\sigma(\mathbb{R})$ is a CW complex if σ is finite-dimensional.

6.3 Local finiteness conditions

The correspondence $\mathbf{Sch}^{\mathrm{aff/qi/nb}}_{\mathbb{F}_1} \to \mathbf{Cone}^N$ has a fully faithful left adjoint

$$(K^\times, \sigma) \mapsto \operatorname{Spec}\mathbb{F}_1[\sigma^\circ \cap K^\times].$$

The counit of this adjunction is an isomorphism - that is, a quasi-integral $\mathbb{F}_1$-algebra A is determined by its cone σ_A - under the following independent conditions:

i) $A \setminus 0$ is a saturated submonoid of $K_A^\times$;

ii) the $\mathbb{Q}_{\geq 0}$-span of $A \setminus 0$ in $K_A^\times \otimes \mathbb{Q}$ is reflexive.

The first condition by definition 5.11 says that $\operatorname{Spec}A$ is *normal*. The second has the following geometric meaning: let $D = (f_+/f_-)$ be a principal divisor; then if the pullback of D along any $\mathbb{Q}$-jet $\Delta_{\mathbb{Q}} \to \operatorname{Spec}A$ is effective, then D is effective. In other words, $\mathbb{Q}$-jets are enough to detect poles of rational functions.

6.2 Definition. Let us say that a quasi-integral scheme X has *enough jets* if a locally principal divisor with effective pullback to any $\mathbb{Q}$-jet is effective.

The left adjoint to Σ associates to an affine quasi-integral scheme X a universal non-boundary morphism

$$X^{\mathrm{ej}} \to X$$

from a normal $\mathbb{F}_1$-scheme with enough jets. This morphism is integral if and only if X already had enough jets, in which case it is the normalisation.

6.3 Lemma. *The adjunction* $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}} \leftrightarrows \mathbf{Cone}^N$ *restricts to an equivalence on the full subcategory of* $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}}$ *whose objects are normal with enough jets.*

6.4 Example. In the case that $\langle\sigma_A\rangle$ is finite-dimensional, A has enough jets if and only if $(A\setminus 0)\otimes\mathbb{Q}\subset K_A^\times\otimes\mathbb{Q}$ is closed in the order topology of $\mathbb{Q}$. For instance, any valuation ring $\mathbb{F}_1[[t^{-H}]]$ with H totally ordered of rank greater than one does not have enough jets. Let $-1\in H$ generate a minimal convex subgroup. Then the reflexivisation of $\mathbb{F}_1[[t^{-H}]]$ is a localisation at t.

If, on the other hand, the topological boundary of $\sigma_A(\mathbb{R})$ has no non-zero rational points, then A automatically has enough jets. Since in this case, the only face is zero, the underlying space of $\operatorname{Spec}A$ is the two-point Sierpinski space: any non-zero function f induces a homeomorphism $\operatorname{Spec}A\to\mathbb{A}^1_{\mathbb{F}_1}$.

Relative finiteness Let us call an object $(\sigma,K^\times)$ of $\mathbf{Cone}^N$ *rational polyhedral* if its span $\langle\sigma\rangle(\mathbb{Q})$ in $N(\mathbb{Q})$ is finite-dimensional and $\sigma\subseteq\langle\sigma\rangle$ is rational polyhedral in the sense of [Ful93, §1.1]. More generally, we say that a morphism $f:\sigma_1\to\sigma_2$ of cones is rational polyhedral if there exists a rational polyhedral cone $\sigma_2'\subseteq N_1$ such that $\sigma_1=\sigma_2'\cap f^{-1}\sigma_2$.

6.5 Lemma. *Let* $A\to B$ *be a non-boundary homomorphism of quasi-integral* $\mathbb{F}_1$*-algebras.*

i) If B *is integral over a finite type* A*-algebra, then the kernel* ker *of* $N_B(\mathbb{Q})\to N_A(\mathbb{Q})$ *is finite-dimensional and* $\sigma_B\to\sigma_A$ *is rational polyhedral.*

ii) If $A\to B$ *is of finite type, then* i) *and* $\ker(\mathbb{Z})\otimes\mathbb{Q}\tilde{\to}\ker(\mathbb{Q})$.

If B *has enough jets, then the converses to these statements hold.*

6.6 Example. Note that $X\to S$ finitely presented does not imply even that $\nu X\to\nu S$ is of finite type. Take, for example, $K^\times=\mathbb{Q}^2$ with basis x,y, and σ_A the positive orthant. The $\mathbb{F}_1$-scheme $X=\operatorname{Spec}A$ is normal; however, the blow-up of X at the finitely generated ideal $T=(x,y)$ is not, and its normalisation is of infinite type. This can be seen from the fact that the convex hull of $T\setminus 0$ in $A\setminus 0$ is not finitely generated as an $A\setminus 0$-module.

6.7 Lemma. *Let* A *be a quasi-integral* $\mathbb{F}_1$*-algebra.*

i) If A *is integral-over-Noetherian, then* σ_A *is rational polyhedral.*

ii) If A *is Noetherian, then* i) *and* $\langle\sigma_A\rangle(\mathbb{Z})\otimes\mathbb{Q}\tilde{\to}\langle\sigma_A\rangle(\mathbb{Q})$.

If A *has enough jets, then the converses to these statements hold.*

Proof. By lemma 2.3, we are reduced to checking finiteness of the homomorphism $\mathbb{F}_1[A^\times]\to A$, which follows from lemma 6.5. □

6.4 Glueing

Since Σ has a left adjoint, it is, in particular, left exact. It is therefore the pullback functor of an essential geometric morphism

$$\phi:\mathrm{PSh}\mathbf{Cone}^N\to\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$$

between the presheaf topoi. We equip $\mathbf{Cone}^N$ with 'open immersions' the class of face inclusions of finite codimension. The category of *embedded cone complexes* - that is, locally representable presheaves on $\mathbf{Cone}^N$ - is denoted $\mathrm{C}\mathbf{Cone}^N$.

As we have already noted, Σ preserves open immersions, and so its extension preserves the subcategories of locally representable objects

$$\Sigma : \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}} \to \mathrm{C}\mathbf{Cone}^N.$$

We thus associate to any quasi-integral scheme $X/\mathbb{F}_1$ a collection of cones glued together along faces in a way that respects their embedding into N.

6.8 Proposition. *The pullback Σ for the geometric morphism*

$$\phi : \mathrm{PSh}\mathbf{Cone}^N \to \mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$$

preserves open immersions and induces bijections on open subobject lattices.

It has a fully faithful left adjoint with image the full subcategory generated under colimits by the normal schemes with enough non-boundary jets.

6.9 Corollary. *The cone complex functor*

$$\Sigma : \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}} \to \mathrm{C}\mathbf{Cone}^N$$

induces, for every $X \in \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$, a homeomorphism $\mathrm{Sh}(\Sigma_X) \tilde{\to} \mathrm{Sh}(X)$.

The left adjoint to Σ is fully faithful with image the category $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{ej/n/nb}}$ of normal schemes with enough jets. In particular, Σ restricts to an equivalence on this subcategory.

6.10 Corollary. *The specialisation order on the topological space underlying a quasi-integral scheme X with enough jets is the inclusion order on cones of Σ_X.*

Developing map By taking colimits, a cone complex Σ can be realised as a functor of pogroups

$$\Sigma_X(H) = \mathrm{Hom}^{\mathrm{nb}}(\Delta_H, X)$$

and, in the case $H = \mathbb{R}$, as an $\mathbb{R}^\times_{\geq 0}$-equivariant topological space. By definition, $\Sigma(\mathbb{R})$ is strongly topologised with respect to cone inclusions $\sigma(\mathbb{R}) \hookrightarrow \Sigma(\mathbb{R})$. The $\mathbb{R}^\times_{\geq 0}$-action contracts $\Sigma_X(\mathbb{R})$ onto a discrete set, which may be identified with $\pi_0(X)$.

If Σ is a *finite* complex - that is, if it is qcqs as an object of $\mathrm{Sh}\mathbf{Cone}^N$ - cone inclusions are actually topological immersions. It follows that on qcqs objects of $\mathrm{C}\mathbf{Poly}^N$, topological realisation is left exact. If the cones of Σ are finite-dimensional, $\Sigma(\mathbb{R})$ is a CW complex. Since every cone complex is a filtered colimit of finite subcomplexes and filtered colimits of CW complexes are exact, topological realisation is left exact on all complexes with finite-dimensional cones.

When $\Sigma \in \mathrm{C}\mathbf{Cone}^N$ is connected, N is constant, and we get a *developing map* $\delta : \Sigma \to N$ that is linear and injective on each cone. If the developing map is globally injective, Σ is nothing more than a *fan* in N.

6.5 Criteria for separation and propriety

Through its embedding into $\mathrm{Sh}\mathbf{Sch}_{\mathbb{F}_1}$, $\mathrm{PSh}\mathbf{Cone}^N$ inherits notions of overconvergence, separation, and propriety of morphisms. These correspond, as in §4, to the class $\mathbb{P}$ of morphisms that give rise under that embedding to integral-over-proper morphisms of schemes.

Suppose that Σ is connected with cocharacter group N. We have seen that the rational (resp. integral) points of N correspond to non-boundary $\tilde{\mathbb{A}}^1_{\mathbb{F}_1} \setminus 0$-points (resp, $\mathbb{A}^1_{\mathbb{F}_1} \setminus 0$-points) of the associated scheme, while rational (resp. integral) points of Σ correspond to $\tilde{\mathbb{A}}^1_{\mathbb{F}_1}$-points (resp. $\mathbb{A}^1_{\mathbb{F}_1}$-points). To every point of N, then, we associate an extension problem which, since $\mathbb{A}^1$ cannot be modified, has a solution if and only if the point lifts to Σ.

In other words, the developing map $\Sigma \to N$ induces an injection (resp. bijection) on $\mathbb{Z}$ or $\mathbb{Q}$-points whenever Σ is separated (resp. overconvergent). More generally, an overconvergent morphism $\tilde{\Sigma} \to \Sigma$ induces Cartesian squares

$$\begin{tikzcd}
\Sigma_X(\mathbb{Z}) \arrow[r] \arrow[d] & N_X(\mathbb{Z}) \arrow[d] & \Sigma_X(\mathbb{Q}) \arrow[r] \arrow[d] & N_X(\mathbb{Q}) \arrow[d] \\
\Sigma_S(\mathbb{Z}) \arrow[r] & N_S(\mathbb{Z}) & \Sigma_S(\mathbb{Q}) \arrow[r] & N_S(\mathbb{Q})
\end{tikzcd}$$

of sets, and a separated morphism at least injects into the fibre product.

Any point $\tilde{\mathbb{A}}^1_{\mathbb{F}_1} \to X$ factors through some quasi-integral closed subscheme. In the spirit of [Ful93, §2.4], it is not unreasonable to expect converse statements. However, since the valuative criterion 4.48 presented in this paper actually depends on the following theorem, the logic here is somewhat inverted.

6.11 Proposition. *Let $f : X \to S$ be a non-boundary morphism of finite type between quasi-integral, locally integral / Noetherian $\mathbb{F}_1$-schemes. The following are equivalent:*

i) f is separated;

ii) f is locally separated;

iii) for each cone σ of Σ_S and connected component Σ_0 of $\sigma \times_{\Sigma_S} \Sigma_X$, $\Sigma_0(\mathbb{Q})$ is a fan in $N_0(\mathbb{Q})$;

Suppose further that either f is finitely presented or that $N_X(\mathbb{Q})$ is locally finite-dimensional.[11] *The following are equivalent:*

i) the restriction of f to each connected component of X is proper;

ii) f is overconvergent;

iii) for each cone σ of Σ_S and connected component Σ_0 of $\sigma \times_{\Sigma_S} \Sigma_X$, $\Sigma_0(\mathbb{R})$ is a finite fan with support $f^{-1}\sigma(\mathbb{R})$.

Proof. These are special cases of theorems 7.19 and 7.23. □

This finishes the proof of theorem 6.1.

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[11] The $N_X(\mathbb{Q})$ finite-dimensional case does not follow from the valuative criterion, but one can obtain it directly from some simple convex geometry arguments.

7 From formal schemes to punctured fans

The straightforward nature of linear topologies on $\mathbb{F}_1$-algebras means that the local picture for integral formal $\mathbb{F}_1$-schemes is largely the same as for algebraic $\mathbb{F}_1$-schemes. However, the global structure is not so trivial as before, and to characterise interesting classes of formal schemes we will be forced to work at the level of cone complexes.

7.1 Diagonalisable groupoid action

7.1 Definition. Let $\mathbf{P}$ be one of the properties quasi-integral, integral, normal. We say that an $\mathbb{F}_1$ formal scheme X is $\mathbf{P}$ if ${}^?X$ is $\mathbf{P}$. A morphism $X \to Y$ of formal schemes is non-boundary if ${}^?X \to {}^?Y$ is non-boundary.

We notate the categories of formal schemes with these properties with the same superscripts as in the algebraic case.

The diagonalisable group action discussed in §6.1 persists for affine, quasi-integral formal schemes, but with no generic point there is no chance of globalising in general.

By patching together the objects $\mathbb{G}_U \times U$ for affine open $U \subseteq X$, we obtain instead an action of a "local system of algebraic groups", or more precisely an $\mathbb{F}_1$-formal *groupoid*

$$\begin{tikzcd}
\mathbb{G}_X^2/X \arrow[r, "\mu"] & \mathbb{G}_X/X \arrow[loop above, "\iota"] \arrow[r, shift left=2, "\pi"] \arrow[r, shift right=2, "\sigma"'] & X \arrow[l]
\end{tikzcd}$$

whose structural morphisms are representable by $\mathbb{F}_1$-schemes. Here of course

$$\mathbb{G}_X/X = \mathrm{Hom}_X(K_X^\times, \mathbb{G}_{m,X}),$$

so the 'character group' of this groupoid is the local system $K_X^\times$.

This structure is functorial for non-boundary morphisms, and a similar story to that of §6.1 also holds for quasi-integral formal subschemes of the boundary.

7.2 Punctured cones

Let $(\mathrm{Spec}\, A, Z) \in {}^Z\mathbf{Sch}_{\mathbb{F}_1}$ be an affine quasi-integral scheme marked along a single subscheme. In §6.2, we associated to A a strongly convex, reflexive cone $\sigma_A \in \mathbf{Cone}^N$ in the rational dual N_A of $K_A^\times$.

The formal completion of A depends only on the complement $\mathrm{Spec}\, A \setminus Z$, which, as discussed in §6.2, corresponds to a union of faces ζ of σ_A. The faces ζ are cut out by the equations $\log f_\zeta = 0$ where $f \neq 0$ is in the ideal of Z. In the setting of lemma 6.3 - that is, when A is normal and has enough jets - $\hat{A}$ is determined by the pair (σ, ζ). These are the combinatorial data we can associate to $\mathbb{F}_1$-formal schemes.

7.2 Definition. An *punctured embedded cone* is a reflexive, strongly convex cone $\sigma \in \mathbf{Cone}^N$ together with a specified collection of proper principal faces $\zeta \subset \sigma$, the *punctures*. If the collection is finite, we say that $\sigma \setminus \zeta$ has *finite puncturing*. A morphism $\sigma_1 \to \sigma_2$ of punctured cones must restrict to a map $\sigma_1 \setminus \zeta_1 \to \sigma_2 \setminus \zeta_2$. The category of punctured embedded cones is denoted $\mathbf{Cone}_*^N$. In the sequel, I will usually omit the word 'embedded', and also the notation ζ, where this cannot cause confusion.

A *face* of a punctured cone is a face of the underlying cone (defined by an element of $K^\times$), not entirely contained in ζ, with the induced puncturing.

Note that the category $\mathbf{Cone}_*^N$ is not finitely complete: faces of a punctured cone can have empty intersection, and the empty set is not a punctured cone.

We have associated a punctured cone to any affine quasi-integral marked scheme *not entirely contained in* Z; that is, any marked scheme in the essential image of the algebraisation functor

$$?:\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}} \to {}^Z\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff/i/nb}},$$

by precomposition with which we obtain a *punctured cone functor*

$$\Sigma:\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}} \to \mathbf{Cone}_*^N, \quad \lim_{n\to\infty} A/T^n \mapsto (\sigma_A,\zeta_A).$$

In the other direction, by considering a punctured cone as a cone and applying the left adjoint construction, we obtain an affine, normal scheme, and the punctures mark it along a closed subscheme. By composing with formal completion, we obtain a fully faithful left adjoint to Σ. Its essential image has objects the normal formal schemes with enough jets.

7.3 Lemma. *The adjunction* $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}} \leftrightarrows \mathbf{Cone}_*^N$ *restricts to an equivalence on the full subcategory of* $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}}$ *whose objects are normal with enough jets.*

7.4 Lemma (Krull intersection)**.** *Let A be a quasi-integral $\mathbb{F}_1$-algebra, $T \trianglelefteq A$ a finitely generated ideal. One of the following are true:*

i) $A \to \hat{A}$ *is injective;*

ii) $\log f:\sigma_A(\mathbb{Q}) \to \mathbb{Q}$ *is identically zero for some* $0 \neq f \in T$.

Proof. Since formal completion along T can be written as a finite sequence of formal completions at the generators of T, we may assume $T=(t)$ is principal. A non-zero element of $\bigcap_{n\in\mathbb{N}}(t^n)$ is, as an additive function $\sigma_A(\mathbb{Q}) \to \mathbb{Q}$, bounded above by all multiples of $\log t$. Such exists if and only if $\log t$ is identically zero. □

7.5 Aside. If A is integral and, say, Noetherian (so $\langle\sigma_A\rangle(\mathbb{Q})$ is finite-dimensional), then this result implies that a formal completion $A \to \hat{A}$ is either bijective or *zero*. If A is only quasi-integral, it may have non-invertible elements that are identically zero on $\sigma_A(\mathbb{Q})$.

Open immersions The stupid feature of monoidal geometry that makes this construction possible is the following:

7.6 Lemma. *The restriction of the functor ? of forgetting the topology to the category of* integral $\mathbb{F}_1$*-algebras preserves localisations.*

Proof. Let A be complete with respect to a finitely generated ideal T, and let $A \to A\{f^{-1}\}$ be a non-zero completed localisation. Then $f \not\in T$ and so, since A is integral, for every $t \in T$, $\log f \neq \log t$. It follows that $\log t$ is also non-zero on the face $\log f = 0$.

By the Krull intersection theorem (lemma 7.4), the T-adic filtration of $A^?[f^{-1}]$ is separated. Therefore $A^?[f^{-1}] = A\{f^{-1}\}^?$. □

This statement is false for quasi-integral algebras. However, since normalisation is a homeomorphism, it still follows that Σ exchanges localisations with face inclusions:

7.7 Lemma. *The punctured cone functor Σ preserves open immersions. For each $A \in \mathrm{Alg}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$, Σ identifies the poset of non-zero localisations of A with that of faces of $\sigma_A \setminus \zeta_A$.*

Points Let us define the H-points of a punctured cone $\sigma\setminus\zeta$ to be $\sigma(H)\setminus\zeta(H)$, so that $\sigma\setminus\zeta(H)$ remains the set of non-boundary morphisms $\Delta_H = \mathrm{Spec}\,\mathbb{F}_1[\![t^{-H}]\!] \to X_\sigma$. It is important to distinguish in the notation between $\sigma\setminus\zeta(H)$ and $\sigma(H)$, and so we will never omit ζ when talking about points.

Topological realisation When $H = \mathbb{R}$, the set of points $\sigma\setminus\zeta(\mathbb{R})$ is a $\mathbb{R}_{>0}^\times$-invariant open subset of a cone in a real vector space. The $\mathbb{R}_{>0}^\times$-action is free as soon as $\zeta \neq \emptyset$. If A is integral-over-Noetherian, then $\sigma_A\setminus\zeta_A(\mathbb{R})$ is a finite-dimensional CW complex.

Relative variant In the case A is defined over a valuation ring $k[\![t^{-H}]\!]$ for some pogroup H and coefficient $\mathbb{F}_1$-field k, we can instead use the punctured cone inside the fibre product

$$\begin{tikzcd}
N_X(H') \arrow[r] \arrow[d] & \mathrm{Hom}(K_X^\times, H') \arrow[d] \\
H' \arrow[r] & \mathrm{Hom}(H, H')
\end{tikzcd}$$

which is defined as a functor of pogroups over H. When H is not a subgroup of $\mathbb{Q}$, the usual cone associated to A will be infinite-dimensional, and so this variant is likely to be more sensible. We use the same notation.

When $H \subseteq \mathbb{R}$, we can also define the topological realisation. If A is integral over a finite type $k[\![t^{-H}]\!]$-algebra, then the variant $\sigma_A\setminus\zeta_A(\mathbb{R})$ is a finite-dimensional CW complex.

7.3 Glueing

The globalisation of punctured cone functor proceeds much as in the schemes case §6.4.

7.8 Lemma. *Σ is a flat functor.*

Proof. Since we are dealing with presheaf categories, we need to show two things: first, that a finite diagram in $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$ that has a limit in $\mathbf{Cone}_*^N$ has a limit in $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$, and second, that Σ is left exact. The latter follows from the fact that Σ has a left adjoint.

A fibre product $\sigma_X \times_{\sigma_Z} \sigma_Y$ is representable in $\mathbf{Cone}_*^N$ if and only if the images in σ_Z of $\sigma_X\setminus\zeta_X$ and $\sigma_Y\setminus\zeta_Y$ have non-empty intersection. In particular, it has a $\mathbb{Q}$-point. The fibre product $X\times_Z Y$ in $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff}}$ is therefore non-empty and hence an object of $\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{aff/qi/nb}}$ (cf. the aside 5.2). ☐

As before, let us define the 'open immersions' in $\mathbf{Cone}_*^N$ to be the principal face inclusions. From lemmas 7.3, 7.7, and 7.8, in reverse order, it follows:

7.9 Proposition. *The punctured cone functor Σ extends to the pullback along a geometric morphism*

$$\mathrm{PSh}\mathbf{Cone}_*^N \to \mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$$

which preserves open immersions and induces bijections on open subobject lattices.

It has a fully faithful left adjoint with image the full subcategory generated under colimits by the normal formal schemes with enough non-boundary jets.

We therefore obtain an adjunction

$$\Sigma : \mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{qi/nb}} \leftrightarrows \mathrm{C}\mathbf{Cone}_*^N$$

between the categories of quasi-integral formal schemes and non-boundary morphisms and that of *punctured cone complexes*, that is, locally representable presheaves on $\mathbf{Cone}_*^N$.

7.10 Corollary. *The cone complex functor and its left adjoint restrict to an equivalence*

$$\Sigma : \mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{ej/n/nb}} \xrightarrow{\sim} \mathrm{C}\mathbf{Cone}_*^N$$

on the category of normal formal $\mathbb{F}_1$-schemes with enough jets.

7.11 Corollary. *The specialisation order on the topological space underlying a quasi-integral formal scheme X with enough jets is the inclusion order on cones of Σ_X.*

Developing map Just as in the unpunctured case, a punctured cone complex can be realised as a functor of pogroups

$$\Sigma_X(H) = \mathrm{Hom}^{\mathrm{nb}}(\hat{\Delta}_H, X)$$

and, by applying it to the case $H = \mathbb{R}$, as an $\mathbb{R}_{>0}^\times$-equivariant topological space. If X is locally integral-over-Noetherian, $\Sigma_X(\mathbb{R})$ is a CW complex. Unlike the scheme case, however, it is not necessarily a 0-type.

The topological realisation commutes with fibre products of

i) finite complexes, and

ii) complexes with finite-dimensional cones.

The cocharacter groups assemble to form a local system N_X on Σ_X. It can also be thought of as a conically invariant local system on $\Sigma_X(\mathbb{R})$. The monodromy of this local system is the obstruction to the developing map $\Sigma_X \to N_X$ being globally defined. In particular, for each choice of cone $\sigma \subset \Sigma_X$ one gets a canonical developing map

$$\delta : \widetilde{\Sigma}_X \to N_{X,\sigma}$$

from the cover $\widetilde{\Sigma}_X$ of Σ_X that trivialises N_X on the component of σ.

The same holds true for the relative situation $\widetilde{\Sigma}_{X/H} \to N_{X/H,\sigma}$.

7.4 Overconvergent neighbourhoods

As in the scheme case, the embedding of $\mathrm{PSh}\mathbf{Cone}_*^N$ into $\mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}^{n/nb}$ equips it with a class of morphisms $\mathbb{P}$ that become formally proper under that embedding. The crux of the study of extension problems will be to identify what birational $\mathbb{P}$ morphisms look like at the level of topological realisations.

Refinements A morphism $\Sigma_X \to \Sigma_Y$ from a finite punctured cone complex to a punctured cone is called a *refinement* if it induces a bijection on points and on $K^\times$. Since taking points commutes with limits, this notion is stable for base change. It therefore makes sense to talk about refinements of arbitrary punctured cone complexes.

One way to induce a refinement of a cone $\sigma \subseteq N$ is by choosing a finite collection of linear functions $\{f_i\}_{i=1}^k \subseteq K^\times$. The functions may as well be assumed non-positive on σ. Thus we reach the following fact, true also in the algebrac case $\mathrm{C}\mathbf{Cone}^N$:

7.12 Lemma. *A refinement induced by a function is in class* $\mathbb{P}$.

Note that the inclusion of any (relatively) polyhedral subcone $\sigma_X \subseteq \sigma_Y$ can be extended to a refinement: if σ_X is defined by equations $\log f_i \leq 0$, then the function you should take is $F_X = 0 \vee \bigvee_{i=1}^k \log f_i$.

We can also define a scaling invariant function $\partial F/\partial r$, where $\partial/\partial r$ generates the $\mathbb{R}_{>0}^\times$-action. Conical compactness of $N_X \setminus 0$ ensures that this function is bounded above.

Puncturing along a subcomplex Let us call an 'open subset' Σ_U of a punctured cone complex Σ_X a *subcomplex*. In other words, a subcomplex is a morphism $\Sigma_U \hookrightarrow \Sigma_X$ whose restriction to each cone of Σ_X is the inclusion of a union of faces, with the induced puncturing.

It makes sense to further puncture Σ_X along a subcomplex Σ_U. At the level of cones $\sigma \setminus \zeta \subseteq \Sigma_X$ it is defined by enlarging the puncturing of σ to $(\Sigma_U \cap \sigma) \cup \zeta$. There is a natural morphism $\Sigma_X \setminus \Sigma_U \to \Sigma_X$.

Equivalently, $\Sigma_U \subseteq \Sigma_X$ gives rise to an open immersion of formal schemes $U \subseteq X$, and $\Sigma_X \setminus \Sigma_U$ is simply the complex obtained by applying Σ to the formal completion of X along the complement of U. In particular:

7.13 Lemma. *The puncturing of a complex along a subcomplex is of class* $\mathbb{P}$.

Note that at the level of topological realisations, a subcomplex whose inclusion is quasi-compact is always *closed*. In particular, this applies to finite subcomplexes of quasi-separated complexes.

7.14 Proposition (Overconvergent n'hoods are open n'hoods)**.** *Let* $\tilde{\sigma}_V \in \mathrm{C}\mathbf{Cone}_*^N$ *be a connected, finite complex,* $\sigma_V \in \mathbf{Cone}_*^N$ *a cone, and* $f : \tilde{\sigma}_V \to \sigma_V$ *a polyhedral morphism fixing a common cell* $\tau \subseteq \sigma_V, \tilde{\sigma}_V$. *The following are equivalent:*

i) $\tilde{\sigma}_V \setminus \tilde{\zeta}_V(\mathbb{R}) \to \sigma_V \setminus \zeta_V(\mathbb{R})$ *is the inclusion of a neighbourhood of* $\tau \setminus \zeta(\mathbb{R})$*;*

ii) f *is an overconvergent neighbourhood of* τ.

Proof. The implication *ii)⇒i)* is the easy direction: it follows from lemmas 7.12 and 7.13 and the fact that a formally projective morphism is, by definition, a composite of a formal completion and an integral-over-projective morphism.

We focus on the converse. Since σ_V and $\tilde{\sigma}_V$ share a face, they also have the same character group. The polyhedral morphism f is therefore obtained by base change from a morphism between rational polyhedral cones in finite-dimensional vector spaces. Since passing to the topological realisation of finite complexes commutes with fibre products, it is safe to work with such a model; we may thus assume that $N_V = N_\tau$ is finite-dimensional and that $\tilde{\sigma}_V$ and σ_V are rational polyhedral. It is now an exercise in elementary convex geometry.

We begin the exercise by forgetting about the punctures. This does not introduce complications because σ_V is a cone and, by assumption, $\tilde{\sigma}_V \setminus \tilde{\zeta}_V \to \sigma_V \setminus \zeta_V$ is injective. Let $\sigma \subseteq \tilde{\sigma}_V$ be a cone defined by the functions $\{f_i\}_{i=1}^k$. Let $p \in N_\tau(\mathbb{R})$ be in the interior of τ. If $\log f_i(p) \leq 0$ for all i, then $p \in \sigma(\mathbb{R})$. Since σ must intersect τ in a face, in this case $\tau \subseteq \sigma$, and

$$F_\sigma := \max\{0, \log f_i\}_{i=1}^k$$

defines a subdivision of σ_V containing both σ and τ as cells.

Otherwise, $\bigvee_{i=1}^k \log f_i$ is strictly positive on the interior of τ. It is therefore dominated by some positive linear combination F of the $\log f_i$. Then the function

$$F_\sigma := \max\{0, \log f_i, F\}_{i=1}^k$$

does the job.

Repeating this choice for each cone of $\tilde{\sigma}_V$, we obtain a finite list of functions F_σ whose join $\bigvee_\sigma F_\sigma$ defines a finite refinement of σ_V that contains τ and contains a refinement of $\tilde{\sigma}_V$ as a subcomplex.

Now let us reintroduce the punctures. Since $\tilde{\sigma}_V \setminus \tilde{\zeta}_V(\mathbb{R})$ is a neighbourhood of $\tau \setminus \zeta(\mathbb{R})$, any cone of σ_U whose interior does not meet $\tilde{\sigma}_V$ is disjoint from $\tau \setminus \zeta$. Let ζ_X be the union of all such cones. Then $\sigma_X \setminus \zeta_X \to \sigma_V \setminus \zeta_V$ is a morphism of class $\mathbb{P}$ that supers $\tilde{\sigma}_V$ and has a section over τ. □

7.15 Corollary (of proof). *Let $f : X \to S$ be a non-boundary morphism of finite type between quasi-integral formal $\mathbb{F}_1$-schemes. If $\Sigma_X \to \Sigma_S$ is a refinement, then f is proper birational.*

7.5 Criteria for overconvergence

Suppose that we are given a morphism $\Sigma_X \to \Sigma_S$ of punctured cone complexes, whose extensional properties we wish to investigate. By lemma 4.13, we are checking extension problems

$$\begin{tikzcd}
\sigma_U \arrow[r] \arrow[d] & \sigma_V \arrow[d] \\
\Sigma_X \arrow[r] & \Sigma_S
\end{tikzcd}$$

with $\sigma_U \subset \sigma_V$ a face inclusion of punctured cones.

The new feature of cone complexes that we will use to understand this extension problem is the developing map $\Sigma_X \xrightarrow{\delta} N_X$. Since this is not everywhere defined, we must first reduce the question to studying a region of Σ_X 'close' to σ_U.

7.16 Definition. Two cones of a punctured cone complex $\Sigma \in \mathbf{CCone}_*^N$ are said to be *contiguous* if they intersect along a principal face. The *big star* of a cone $\sigma \subset \Sigma$ is the complex $\sigma^\star$ obtained by puncturing Σ along all cones discontiguous with σ. The *small star* is the subcomplex $\sigma^* \subseteq \sigma^\star$ of cones containing σ as a principal face.

A punctured cone complex is said to be *locally finite* if the star of any cone is finite and has finite puncturing.

The developing map of a star-shaped complex

$$\sigma^\star \xrightarrow{\delta} N_{\sigma^\star} = N_{X,\sigma}$$

is globally defined. Note that if Σ is a connected cone complex without punctures, then all cones are pairwise contiguous, and so the star of any cone of Σ is equal to Σ itself.

7.17 Lemma. *Suppose that Σ has finite puncture type.*

i) The big star $\sigma^\star(\mathbb{R})$ is a neighbourhood of $\sigma(\mathbb{R})$ in $\Sigma(\mathbb{R})$.

ii) The small star $\sigma^*(\mathbb{R})$ is a neighbourhood of $\sigma(\mathbb{R})$ minus its proper faces in $\Sigma(\mathbb{R})$.

Proof. Indeed, since $\Sigma(\mathbb{R})$ is strongly topologised with respect to cone inclusions, it is enough to check these statements on every cone. □

Geometrically, the open star of $\sigma_U \subseteq \Sigma_X$ is obtained by formally completing X along the closure of U. In the language of §3.1, it is the *formally embedded closure* $\mathrm{cl}(U/X)$ of U.

7.18 Lemma. *$\sigma^\star \to \Sigma_X$ is an overconvergent neighbourhood of σ/Σ_X.*

In other words, to check overconvergence of a morphism $\Sigma_X \to \Sigma_S$ near $\sigma \subseteq \Sigma_X$, it is necessary and sufficient that its restriction to $\sigma^\star$ be overconvergent.

Separated Suppose that we have an extension problem σ_U/σ_V and that σ_U lands in a cone $\sigma_0 \subseteq \Sigma_X$. By lemma 7.18, we may, without loss of generality, replace Σ_X with the star of σ_0. We obtain a linear map $\varphi : N_U \to N_{X,0} = N_{\sigma_0}$ and, since $N_U = N_V$, a commuting diagram

$$\begin{tikzcd}
\sigma_U \arrow[r] \arrow[d] & \sigma_V \arrow[r] & N_U \arrow[d, "\varphi"] \\
\sigma_0 \arrow[r] & \sigma_0^\star \arrow[r, "\delta"] & N_{X,0}
\end{tikzcd}$$

By the characterisation 7.14 of overconvergent neighbourhoods, solutions $\mathrm{Sur}_{U/V} \to \sigma_0^\star$ correspond to lifts of $\varphi : \sigma_V(\mathbb{R}) \to N_{X,0}(\mathbb{R})$ along δ in a neighbourhood of $\sigma_U(\mathbb{R})$. Uniqueness of solutions is therefore related to local injectivity of the developing map.

7.19 Theorem. *Let $\Sigma_X \in \mathbf{CCone}_*^N$. The following are equivalent:*

i) Σ_X is locally separated;

ii) Σ_X is separated and any contiguous pair of cones intersect in a face;

iii) the restriction of the developing map to the union of any contiguous pair of cones is injective;

iv) the restriction of the developing map to the small star of any cone is injective.

Proof. The equivalence *iii)⇔iv)* and the implication *ii)⇒i)* are straightforward.

Suppose *iii)*, and let $\sigma_V \rightrightarrows \sigma_0^\star$ be a pair of solutions. Since the homomorphism of character groups is predetermined, it suffices to check on points.

We may assume σ_V is a single cone, in which case both maps factor through the union of two cones $\sigma_1, \sigma_2 \subseteq \sigma_0^\star$ contiguous along a face containing the image of σ_U. Both solutions must lift along the developing map, which by *iii)* is injective. Therefore they are equal.

Criterion *iii)* also implies that contiguous cones must intersect in at most one face. In particular, Σ_X is quasi-separated, and *ii)* follows.

It remains to prove *i)⇒iii)*. Write

$$\sigma_U := \sigma_1 \cap \sigma_2 \subseteq \Sigma_X, \quad \sigma_V := \delta(\sigma_1) \cap \delta(\sigma_2) \subseteq N_U$$

and consider the extension problem σ_U/σ_V. By proposition 7.14, local separatedness implies that the two sections $\sigma_V(\mathbb{R}) \to \sigma_i(\mathbb{R})$ must be equal in a neighbourhood of $\sigma_U(\mathbb{R})$. Therefore either $\sigma_1 = \sigma_2$ or $\sigma_V = \sigma_U$. It follows that the restriction of δ to $\sigma_1 \cup \sigma_2$ is injective. □

Intuitively, separatedness of Σ_X means that the developing map cannot 'double back' when passing through from one cone into a neighbouring cone.

Suppose that Σ_X is separated and locally finite-dimensional. For any point p of $\Sigma_X(\mathbb{R})$, there is a unique minimal cone σ containing p. The small star $\sigma^*(\mathbb{R})$ of σ is then a neighbourhood of p in $\Sigma_X(\mathbb{R})$. By proposition 7.19, the restriction of δ to $\sigma^*(\mathbb{R})$ is injective. In other words, the developing map is a *local immersion*.

7.20 Corollary. *Suppose that every cone of Σ_X is finite-dimensional. Then Σ_X is separated if and only if the developing map is a local immersion.*

If Σ_X is also locally finite, then in fact every cone has a neighbourhood in $\Sigma_X(\mathbb{R})$ on which the developing map is an immersion.

Proof. The last statement is, *a priori*, a bit stronger than what we discussed. Since we won't use that result, I only provide a sketch of the proof.

Suppose Σ_X is locally finite and separated, and let σ be a cone. We will construct a set $U \subseteq N_\sigma(\mathbb{R})$ whose preimage in $\sigma^\star(\mathbb{R})$ is a connected neighbourhood of $\sigma \setminus \zeta(\mathbb{R})$ and such that δ induces a bijection

$$\pi_0(\sigma^\star \setminus \sigma) \to \pi_0(U \setminus \sigma(\mathbb{R}))$$

and is therefore injective for topological reasons. First, by finiteness of the punctures there exists a (not necessarily strongly) convex polyhedral cone $C \subseteq N_\sigma(\mathbb{R})$ containing σ and such that $\partial C \cap \sigma = \zeta$. Choose a deformation retract r of the interior of C onto $\sigma \setminus \zeta(\mathbb{R})$.

Let T denote the union of all punctured faces of σ contiguous with another cone. The connected components of T are indexed by $\pi_0(\sigma^\star \setminus \sigma)$. Setting $U = r^{-1}T \cup \sigma \setminus \zeta(\mathbb{R})$, we have $U \cap \sigma = T$ and so $r : U \setminus \sigma \to T$ is a weak equivalence. □

7.21 Corollary. *Let $f : X \to S$ be a non-boundary morphism of quasi-integral formal schemes with enough jets. The following are equivalent:*

i) f is locally separated;

ii) f is separated and has affine diagonal;

iii) for each cone σ_S of Σ_S and σ_X of Σ_X, the restriction of the developing map to $f^{-1}\sigma_S \cap \sigma_X^*$ is injective;

and in the case that $\Sigma_X(\mathbb{R})$ (or $\Sigma_{X/H}(\mathbb{R})$) is locally finite dimensional, we may add

iv) over each cone of Σ_S, the developing map is a local immersion.

Overconvergent We return to the situation of an arbitrary morphism $f : \Sigma_X \to \Sigma_S$ of cone complexes and extension problem σ_U/σ_V. We are trying to complete the square

$$\begin{tikzcd}
\sigma_V \arrow[r] \arrow[d, dashed] & N_U \arrow[d, "\varphi"] \\
\sigma_0^\star \arrow[r] & N_{X,0}
\end{tikzcd}$$

for a cone $\sigma_0 \subseteq \Sigma_X$ containing the image of σ_U. Evidently, any solution must factor through the star of σ_U in the 'fibre product'

$$\tilde{\sigma}_V \subseteq \sigma_V \times_{N_{X,0}} \sigma_0^\star := \operatorname*{colim}_{\sigma \subseteq \sigma_0^\star}(\varphi^{-1}\sigma \cap \sigma_V).$$

In other words, the extension problem has a solution if and only if $\tilde{\sigma}_V \to \sigma_V$ is an overconvergent neighbourhood of σ_U.

Applying this to the case $\sigma_U = \sigma_0$, we find:

7.22 Lemma. *Let $f : \Sigma_X \to \Sigma_S$ be overconvergent, and suppose that for every cone σ_S of Σ_S and σ_X of Σ_X, $f^{-1}\sigma_S \cap \sigma_X^\star$ is finite-dimensional. Then f is locally finite.*

Proof. In this case every strongly convex cone in N_U containing σ_U as a face must intersect $\sigma_U^\star$ in finitely many cones. If $\delta(\sigma_U^\star)$ is contained in a finite-dimensional subspace, then this is enough to conclude that it is finite. □

Let us call the finiteness condition of the lemma *locally finite-dimensional*. If f is locally finite, then it is equivalent to the condition that every cone of Σ_X be finite-dimensional. We henceforth assume both of these conditions.[12]

7.23 Theorem. *Suppose that $\Sigma_X(\mathbb{R})$ (or $\Sigma_{X/H}(\mathbb{R})$) is locally finite-dimensional, and let $f : \Sigma_X \to \Sigma_S$ be a locally polyhedral morphism. The following are equivalent:*

i) f is overconvergent;

ii) f is locally finite and the developing map is a local homeomorphism.

In this case, $N_{X/S}$ is everywhere spanned by cones of Σ_X and in particular, finite-dimensional.

Proof. Let σ_U be a cone of Σ_X. Overconvergence says that for any cone σ_V in $f^{-1}\sigma_S$ containing σ_U as a principal face, $\sigma_V \cap \sigma_U^\star$ is a neighbourhood of $\sigma_U(\mathbb{R})$. Thus in any finite-dimensional subspace H of $N_X(\mathbb{R})$, $\sigma_U^\star(\mathbb{R})$ is a neighbourhood of $\sigma_U(\mathbb{R})$ in $H \cap f^{-1}\sigma_S$. The finiteness of $\sigma_U^\star$ implies therefore that $f^{-1}\sigma_S$ is finite-dimensional.

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[12] In fact, for certain questions the locally finite case can be reduced to the finite-dimensional case by picking a model of the star; however, for want of applications, we do not ask these.

Assume $\Sigma_S = \sigma_S$ consists of a single cone. Let σ_0 be a cone of Σ_X. By local finiteness, we may replace $\sigma_0^\star \to \sigma_S$ with a polyhedral model. We will prove that this is overconvergent near σ.

Let σ_U/σ_V be an extension problem, and let $\tilde{\sigma}_V$ be the 'canonical solution'. By hypothesis, there is a polyhedral open neighbourhood of σ_0 in $\sigma_0^\star$ the restriction to which of δ is an open immersion. The inclusion of this neighbourhood is overconvergent at σ_0. Therefore its pullback to $\tilde{\sigma}_V$ is an overconvergent neighbourhood of σ_U in σ_V. □

7.24 Corollary. *Let $f : X \to S$ be a non-boundary morphism locally of finite type between quasi-integral formal schemes with enough jets. Suppose that S is locally integral-over-Noetherian (resp. integral over finite type over a valuation ring with value group $H \subseteq \mathbb{R}$).*

The following are equivalent:

i) f is overconvergent;

ii) f is paracompact and over each cone of Σ_S, the developing map of Σ_X (resp. $\Sigma_{X/H}$) is a local homeomorphism.

In this case, the developing maps equip $\Sigma_X(\mathbb{R})$ with the structure of a $\mathbb{R}^\times_{>0}$-equivariant smooth manifold - in fact, an *affine manifold* - whose boundary is vertical over $\Sigma_S(\mathbb{R})$. In particular, the fibres of f are manifolds without boundary. If f is proper, then the fibres are compact.

7.25 Corollary. *The fibres of an overconvergent (resp. proper) morphism $\Sigma_X(\mathbb{R}) \to \Sigma_S(\mathbb{R})$ are affine manifolds (resp. conically compact affine manifolds) without boundary in a unique way such that the relative developing map $f^{-1}(p) \xrightarrow{\delta} N_{X/S}(\mathbb{R})$ is a local affine diffeomorphism.*

7.6 Meromorphic functions as obstruction to algebraisation

The meromorphic function sheaf of any connected, quasi-integral $\mathbb{F}_1$-scheme is constant. A necessary condition for algebraisability of a connected, quasi-integral formal $\mathbb{F}_1$-scheme X is therefore that $K_X^\times$ be constant.

If this is the case, then N_X is defined as a rational vector space, and we get a global developing map

$$\delta : \Sigma_X \to N_X.$$

We can use δ to 'put back in' the intersections of open sets in an algebraisation of X that became empty upon formal completion. Informally, the prescription for two punctured cones $\sigma_i \setminus \zeta_i$ in Σ_X is:

$$\bigcap_{i=1}^k \sigma_i := \begin{cases} \{0\} & \text{if } \bigcap_{i=1}^k (\sigma_i \setminus \zeta_i) = \emptyset \\ \delta\left(\bigcap_{i=1}^k \sigma_i\right) & \text{otherwise} \end{cases}$$

Note that unless X has affine diagonal, $\bigcap_{i=1}^k \sigma_i$ might consist of several faces. This defines an atlas for an object ${}^?\Sigma_X$ of PSh$\mathbf{Cone}^N$ whose transition maps are principal face inclusions. Moreover, it comes equipped with a compatible collection of marked cones ζ that, when punctured, retrieve Σ_X.

To see that it is locally representable, we can apply the following elementary lemma:

7.26 Lemma. *Let* $\mathrm{Sh}\mathbf{C}$ *be a spatial theory,* I *a cofiltered poset, and* $V : I \to \mathbf{C}$ *a functor such that for every* $f \in I$, Vf *is an open immersion. Suppose that* V *is* locally flat, *meaning that for each* $i \in I$, $V : I_{/i} \to \mathbf{C}_{/V_i}$ *is flat. Then* V *is an atlas for a locally representable sheaf; that is,* $\operatorname{colim} V$ *is locally representable and* V *is an open covering.*

Proof. By proposition 1.2, the condition on the slice sets ensures that for any $i \in I$ we get a geometric morphism $\mathrm{Sh}\mathbf{C}_{V_i} \to \mathrm{Sh}I_i$ whose pullback takes monomorphisms to open immersions. It follows that for any subset K of $I_{/i}$, the restriction $V_{/K}$ of V to the slice set $I_{/K}$ is an open subset of V_i.

Since $\mathrm{Sh}\mathbf{C}$ is a topos, a pushout of monomorphisms is also a pullback. In particular, a pushout of two locally representable sheaves along an open subset is locally representable.

By induction on the number of elements, $V_{/J}$ is locally representable for any finite $J \subseteq I$, and an inclusion $J \subseteq K$ induces an open immersion $V_{/J} \hookrightarrow V_{/K}$. The result now follows from the fact that inductive systems of open immersions are locally representable. □

It follows that ${}^?\Sigma_X$ is a cone complex that can be punctured to retrieve Σ_X. If X is normal with enough jets, this of course yields an algebraisation of X itself.

We make these arguments precise in the proof of the following statement, which does not, in fact, depend on the classification theorem:

7.27 Theorem. *Let* X *be a connected, integral formal scheme. The following are equivalent:*

i) X *is algebraisable;*

ii) $K_X^\times$ *is a constant sheaf.*

There exists an algebraisation functor

$$?:\mathbf{FSch}^{\mathrm{alg/i/nb}}_{\mathbb{F}_1} \to {}^Z\mathbf{Sch}^{\mathrm{i/nb}}_{\mathbb{F}_1}, \quad X \mapsto {}^?X$$

on the category of algebraisable integral formal schemes such that ${}^?X$ *has affine diagonal over* $\mathbb{F}_1$ *whenever* X *does.*[13]

Proof. The restriction of the forgetful functor to $\mathbf{FSch}^{\mathrm{aff/i/nb}}_{\mathbb{F}_1}$ is left exact; it follows that the restrictions of

$$?:\mathcal{U}^{\mathrm{aff}}_{/X} \to \mathbf{Sch}^{\mathrm{aff}}_{\mathbb{F}_1},$$

to slice sets are flat, *except* possibly for fibre products realised by $\emptyset \not\in \mathbf{FSch}^{\mathrm{aff/i/nb}}_{\mathbb{F}_1}$. We will need to tweak $\mathcal{U}^{\mathrm{aff}}_{/X}$ to correct for this. The tweaks may be separated into steps.

Step I Write $\mathcal{U}^{\mathrm{aff/in}}_{/X} \subset \mathcal{U}^{\mathrm{aff}}_{/X}$ for the subcategory of inhabited affine subsets of X. The restriction of ? to $\mathcal{U}^{\mathrm{aff/in}}_{/X}$ preserves fibre products and take all arrows to open immersions, but we will need to introduce new limits in order to get flatness.

---

[13] Note that the corresponding statement for formal algebraic spaces is completely trivial.

Step II The universal way to extend ? to a locally flat functor is to enlarge $\mathscr{U}_{/X}^{\mathrm{aff/in}}$ to the poset $\widetilde{\mathscr{U}}_{/X}^{\mathrm{aff}}$ of non-empty finite subsets of $\mathscr{U}_{/X}^{\mathrm{aff/in}}$ closed under fibre products. This poset has all fibre products, and the left Kan extension

$$\mathrm{LKE} : \widetilde{\mathscr{U}}_{/X}^{\mathrm{aff}} \to \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff}}, \quad \{U_i\}_{i\in I} \mapsto \lim_{i\in I} U_i$$

preserves these. Therefore LKE is locally flat. However, the new morphisms introduced between disjoint families $\{U_i\}$ are not open immersions.

Step III Since $K_X^\times$ is a constant sheaf, we have a map

$$\mathbb{G}_X = \operatorname{Spec} K_X \to {}^?X$$

that factors through LKE. Since $\mathbb{G}_X$ is a single point, $\mathbb{G}_X \to \mathrm{LKE}$ is an affine morphism. We will replace LKE with the embedded closure of $\mathbb{G}_X$, which is calculated as

$$\{U_i\}_{i=1}^k \mapsto \operatorname{Spec}\left(\prod_{i=\in I} \mathscr{O}(U_i) \subseteq K_X \right).$$

The resulting functor $\widetilde{\mathscr{U}}_{/X}^{\mathrm{aff}} \to \mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{aff}}$ preserves fibre products and open immersions. Therefore by lemma 7.26, it is an atlas for ${}^?X$, and ${}^?X$ is a scheme. □

Let X be an integral formal scheme, and let $x \in X(\mathbb{F}_1)$. There is an action

$$\pi_1(X, x) \to \operatorname{Aut}(K_{X,x}^\times)$$

which obstructs algebraisation. Since $\Sigma_X(\mathbb{R})$ has the same weak homotopy type as X, this can also be thought of as an action of $\pi_1(\Sigma_X(\mathbb{R}), \sigma_x)$ with σ_x the cone corresponding under corollary 7.11 to the point x. The kernel of the action corresponds to a covering space of X on which $K_X^\times$ is trivialised. In particular,

7.28 Corollary. *Every integral formal scheme admits an algebraisable covering space.*

7.29 Aside. Separatedness of a formal scheme by no means implies that its algebraisation will be separated. Indeed, any covering space of an algebraisable integral formal scheme is algebraisable, but the algebraisation will be separated if and only if the covering is trivial.

Part II

Analytic geometry

8 Introduction

The idea that degenerations of complex manifolds can be studied using torus fibrations over an affine manifold originates in the work of Hitchin [Hit97] and the SYZ conjecture [SYZ96] in mirror symmetry. In [GS03], the authors of the Gross-Siebert programme made this notion precise using logarithmic geometry.

The purpose of this part is to begin the development of the same ideas instead in the setting of *non-Archimedean* geometry, as proposed in [KS06]. Our approach will be, in continuation of the methods of part I, to define a version of rigid analytic geometry purely in terms of multiplicative monoids. The punchline of the paper is that the resulting category actually *includes* the category of affine manifolds; the monomials in the co-ordinate monoids are, symbolically, the exponentials of the affine functions.

In fact, the Raynaud-style approach to rigid analytic geometry taken here is sufficiently modular that the theory we obtain is strictly a generalisation of ordinary rigid analytic geometry over $\mathbb{Z}$: the latter can be recovered by simply plugging in the category of rings where, in this paper, we put the category of monoids with zero. The same is true for the theory of overconvergence introduced in part I and continued here. For a more detailed discussion of these features, see §8.1 and §1.2.

This modularity also allows us to obtain a family of well-behaved base change functors from the category of analytic spaces over a 'valuation $\mathbb{F}_1$-field' to any topological field, parametrised by the open unit disc of that field.

Collages

Let $\Delta \subseteq N$ be a rational, strongly convex polyhedron in a $\mathbb{Z}$-affine space. In §9, we make the elementary observation that the structures carried by the monoid $\mathrm{Aff}_\Delta(N,\mathbb{Z})$ of affine functions on N that are bounded above on Δ are essentially the data of a *normal Banach algebra* $\mathbb{F}_1(\!(t)\!)\{\Delta\}$ of finite type over the discrete valuation $\mathbb{F}_1$-field $\mathbb{F}_1(\!(t)\!)$.

This produces an invertible correspondence

$$\{\text{polyhedra}\} \quad \leftrightarrow \quad \{\text{normal affine rigid analytic spaces of finite type over } \mathbb{F}_1(\!(t)\!)\}.$$

After base changing to a non-Archimedean field K with uniformiser t, we obtain a rigid analytic space $\mathrm{Spec}\,K\{\Delta\}$ which, in usual terminology, is a *rational domain* in affine space with polyhedron of convergence given by Δ. It also carries a natural action by a unitary group $\mathrm{U}_1 \otimes N$.

Any normal analytic space, locally of finite type over $\mathbb{F}_1(\!(t)\!)$ is therefore obtained by glueing together the spectra of various algebras of the form $\mathbb{F}_1(\!(t)\!)\{\Delta\}$. The global combinatorial object must be obtained by glueing together overlapping embedded polyhedra. I have called these objects *collages* in embedded polyhedra. The immediate conclusion is then (cf. Corollary 7.10):

Theorem (12.10)**.** *The convergence complex contruction induces an equivalence*

$$\Delta_{-/\mathbb{Z}} : \mathbf{Rig}^{\mathrm{ltf/n/nb}}_{\mathbb{F}_1(\!(t)\!)} \xrightarrow{\sim} \mathrm{C}\mathbf{Poly}^{N}_{\mathbb{Z}}$$

between the category of normal rigid analytic spaces locally of finite type over $\mathbb{F}_1((t))$ and the category of collages in embedded lattice polyhedra.

A family of open subsets $U_i \subseteq X$ is a covering if and only if on every polyhedron Δ of $\Delta_{X/\mathbb{Z}}$ there is a finite refinement such that $\Delta(\mathbb{Q}) = \bigcup_i \Delta \cap \Delta_{U_i/\mathbb{Z}}(\mathbb{Q})$.

A $\mathbb{Z}$-affine manifold has a natural notion of embedded lattice polyhedron. Using this, one can easily realise affine manifolds as particularly nice collages, and therefore as rigid analytic spaces. Thus we reach the title result of the series:

Theorem (12.26)**.** *The category of $\mathbb{Z}$-affine manifolds embeds as the full subcategory of the category of rigid analytic spaces over $\mathbb{F}_1((t))$ whose objects are boundaryless, overconvergent, and locally of finite type. Affine open subsets of the rigid space correspond to compact polyhedra inside the affine manifold.*

Much like for punctured cone complexes, there is also a natural notion of developing map $\delta : \Delta \to N$ for collages. The new feature here is that N, being an affine space, is in particular a collage, and the developing map is actually a local immersion of collages. If Δ is also an affine manifold, then this recovers the classical notion of development.

It is not too hard to combinatorially classify the points of the analytic spaces associated to polyhedra, and hence, by extension, collages. The classification involves a division into 'types' generalising Berkovich's language for describing the points of non-Archimedean curves. The calculation can be found in the appendix A.

Overconvergence

Passing from the world of real or complex analytic geometry to that of rigid analytic geometry, one quickly encounters an alarming profligation of new phenomena: even under local finiteness conditions, rigid analytic spaces can manifest all kinds of pathological topological features.

Remarkably, a single extra stipulation - that of *overconvergence* - when applied everywhere, simplifies matters to the point that for many purposes, we may pretend we are once again doing complex topology. For example, Deligne [Del92] introduced the notion of overconvergent open immersion to recover Berkovich's Hausdorff topology on a rigid analytic space. This definition was later generalised to arbitrary morphisms, under the name 'partially proper', by Huber [Hub96, §8]. Overconvergence, in a slightly different guise, is also essential to the study of p-adic cohomology of non-compact geometries.

On a slightly more down-to-earth level, in the first part we saw that the overconvergence condition on a formal scheme implies that the parametrising punctured cone complex is actually a *manifold* without boundary. The same logic applies to collages: overconvergence over $\mathbb{F}_1((t))$ forces the real points of the collage to be an affine manifold, and this can be used to recover theorem 12.26.

However, in the analytic regime we even have a slightly more direct route available: following Deligne, we may re-topologise X using only the overconvergent open immersions to obtain, under fairly general circumstances, a Hausdorff topological space X^{sur} and universal separation map

$$b : X \to X^{\mathrm{sur}}.$$

Given its definition, it can hardly be surprising that the properties of X^{sur} are related to absolute overconvergence of X.

The non-trivial - though by no means difficult to prove - observation is that the topological realisation of the collage associated to X is then actually *homeomorphic* with X^{sur}. In light of this fact, the arguments leading up to theorem 12.26 become rather tautological.

Theorem (12.12)**.** *Let X be a normal rigid space, locally of finite type over $\mathbb{F}_1((t))$, with associated collage Δ_X. There is a natural map $\Delta_X(\mathbb{R}_\infty) \to X$, and the composition*

$$\Delta_X(\mathbb{R}_\infty) \to X \to X^{\mathrm{sur}}$$

is a homeomorphism.

In particular, if X is overconvergent, then X^{sur} is actually a manifold. In other words, starting from an algebraically-defined category and applying principles of pure rigid analytic geometry, we obtain a class of Hausdorff topological manifolds that has been studied by manifold topologists since time immemorial [GH84b, FG83].

Well-understood principles that govern the latter may therefore shed light on the former (and, perhaps, vice versa). For instance, compact affine manifolds that are *complete* - a condition conjecturally equivalent to a Calabi-Yau property - have been classified in dimensions up to three.

Finally, in §13 we produce a base change from the overconvergent site of rigid analytic spaces over $\mathbb{F}_1((t))$ to the category of complex analytic spaces that recovers the classical construction

$$\mu : TB/\Lambda^\vee \to B$$

of torus fibrations over affine manifolds. For the simple geometries that this statement concerns, it makes precise the idea that overconvergent topology 'looks like' the topology of a complex analytic space.

B-model torus fibration

Many ideas here have been motivated, if indirectly, by the mirror dual construction in symplectic geometry, features of which I expect will continue to inspire future work.

The starting point is the symplectic theory of toric manifolds, which revolves around the Delzant construction. This construction parametrises a symplectic manifold (X,ω) with a Hamiltonian action of a compact torus T in terms of a completely integrable system

$$X \to \Delta$$

with Δ a polyhedron inside an affine space modelled on the dual Lie algebra $\mathfrak{t}^\vee$ of T. One can pass to a 'large radius limit' in which Δ is a partial compactification of $\mathfrak{t}^\vee$. If X carries a complex structure, then this large polyhedron is controlled by the fan of X - alternatively, it is another avatar for the $\mathbb{F}_1$-structure given by the torus embedding.

A more general situation that arises in SYZ mirror symmetry is when X has the structure of a Lagrangian torus fibration over a manifold B. The T-action and embedding into an affine space under $\mathfrak{t}^\vee$ are now only defined locally on B. Globally, the base attains a reduction of structure group to $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$, making it into an $\mathbb{R}$-*affine manifold*. With a rational symplectic form and choice of pre-quantum structure on X, this can be refined to a $\mathbb{Q}$-affine structure by considering the 'Bohr-Sommerfeld' Lagrangians.

The SYZ conjecture predicts that via a ‘Legendre dual’ construction, B can also be thought of as parametrising a certain maximal degeneration of Calabi-Yau varieties, and in particular, a rigid analytic space $X^\vee$ over $\mathbb{C}((t))$. The modern form of this conjecture has this dual ‘parametrisation’ a kind of *non-Archimedean torus fibration*

$$\mu : X^\vee \to B$$

as in [KS06] (where the concept is made precise using the Berkovich visualisation of $X^\vee$). This fibration carries a locally defined action by the dual torus with Lie algebra $\mathfrak{t}^\vee$.

The thesis of this work is that non-Archimedean torus fibrations are the natural generalisation of toric geometry to the rigid analytic world. Though we focussed on $\mathbb{F}_1$ in this paper, I will return to the geometry of $X^\vee$ itself in a future work.

By further analogy with the symplectic story, one might also imagine generalisations to integrable systems with singularities; this would allow us to study a much larger class of analytic spaces. Indeed, certain singularities of ‘focus-focus’ type are already allowed in the Gross-Siebert programme, which is concerned with fairly general maximally degenerate Calabi-Yau manifolds. However, a discussion of these ideas is far beyond the scope of the present work.

8.1 On defining rigid analytic geometry

According to Raynaud [Ray74], a rigid analytic space is what you get when you puncture a formal scheme along a closed subscheme. Following this principle, we define rigid analytic geometry by a certain localisation procedure applied to a category ${}_Z\mathbf{FSch}$ of formal schemes marked with a family of closed subschemes Z that was discussed in §I.2.6.

The morphisms that get inverted are, intuitively, those birational maps that are isomorphisms ‘away from Z’. Such maps are generated by what are usually termed ‘admissible’ modifications. Our procedure §10.1 differs from simply inverting these morphisms (like in [Ray74]) only in that we force it to be compatible with *glueing*; that is, with the structure of what will be the *rigid topos* Sh**Rig**. Literally, all this means is that our localisation functor

$$\mathrm{Sh}_Z\mathbf{FSch} \to \mathrm{Sh}\mathbf{Rig}$$

preserves *colimits*. This localisation procedure is almost identical to that of [FK13], though I have plumped for a more standard language to describe it.

On the other hand, our input category ${}_Z\mathbf{FSch}$ is a generalisation of that considered in *op. cit.*, the primary improvements it provides being that **Rig** contains the category of formal schemes, and that the map

$$j : X \to X^+$$

exhibiting X^+ as a formal model of X is actually a morphism of rigid spaces. Analytic spaces in the traditional sense - that is, for which Z is locally a reduction of a formal model - are called *purely* analytic.

One can reproduce many of the basic definitions and arguments of rigid geometry by means of a ‘lift’ to ${}_Z\mathbf{FSch}$ (§10.2).

The Raynaud-style definition has the advantage of clearly producing the category of objects we want to study, but it lacks a certain concreteness. It will also be useful to have local descriptions of rigid analytic spaces in terms of *topological commutative algebra*; in particular, this will be essential to get any kind of module theory (though we don't pursue that in this paper). I only discuss topological $\mathbb{F}_1$-algebras, considering the case of commutative rings to have been established already.

This local algebra is where the $\mathbb{F}_1$-regime enjoys considerable simplifications relative to the $\mathbb{Z}$-regime, the main point of departure being that as soon as a module is *Hausdorff*, it is *complete*. We are therefore able to switch between a point-set-topological and pro-object description of topological $\mathbb{F}_1$-algebras and modules.[14] In §10.3, I review the various elementary assumptions one needs to get the theory off the ground, along with some remarks their geometric provenance. One arrives at a certain category of 'Tate' algebras, that is, pairs $(A;A^+)$ consisting of a pro-discrete ring A^+ and a localisation A thereof.

The spectrum of A^+ is a formal scheme, and one marks the divisors cut out by the functions inverted by the localisation $A^+ \to A$. The rigid space obtained by puncturing this marking is labelled $\operatorname{Spec} A$ (§10.4). This produces a functor

$$\operatorname{Spec} : \{\text{topological rings}\} \to \mathbf{Rig}.$$

The only thing remaining to recover the picture of 'affine objects' familiar from algebraic geometry is to invert the construction. It turns out that this is not quite possible: there are non-trivial admissible modifications between affine marked formal schemes.

We can salvage the situation by identifying exactly which morphisms are inverted at the level of algebra. The algebraic detail you have to know to get this to work is:

> What happens to $\Gamma\mathcal{O}_{X^+}$ under an admissible blow-up (of a model X^+ of $\operatorname{Spec} A$)?

By finiteness of global sections over projective morphisms, the answer is that you get an *integral algebra extension* inside A (§I.3.5). Thus the full, localising subcategory of pairs $(A;A^+)$ such that A^+ is *integrally closed* inside A embeds fully faithfully via Spec into **Rig**.

General nonsense (cf. §I.1.3) also provides us with an underlying topological space for our rigid analytic spaces. This is the famous *Riemann-Zariski space*, defined here §10.6 as a limit over all formal models (with a more explicit description, in a special case, in the appendix A). If one didn't already know to look at this space, one could still find its definition by inspecting the nuts and bolts of the localisation construction.

9 Remarks on polyhedra

Let $H \subseteq \mathbb{R}$ be a totally ordered additive group, and write $H^\circ := H \cap \mathbb{R}_{\leq 0}$. An *$H$-rational polyhedron* is a subset of an H-affine space N defined by a finite list of inequalities with coefficients in H.

We will be interested in the filtered monoids that arise as sets of affine functions bounded above on a polyhedron, and how they can be used to recover the combinatorial structure of the same polyhedra.

Let N be an affine space over H, and let $\Delta \subseteq N$ be an H-rational polyhedron. We introduce the partially ordered monoids

$$\mathrm{Aff}^+_\Delta(N,H) \quad \subseteq \quad \mathrm{Aff}_\Delta(N,H) \quad \subseteq \quad \mathrm{Aff}(N,H)$$

---

[14] In general, the former description breaks down over $\mathbb{Z}$.

of, in reverse order, affine functions on N with integral slopes, affine functions *bounded above* on Δ, and affine functions bounded above by zero on Δ. For the moment, I will be deliberately vague about what kind of objects N and Δ are, for the most part considering them as formally dual to the partially ordered monoids in which they are encoded.

9.1 The ambient affine space

The monoid $\mathrm{Aff}(N,H)$ is actually a torsion-free Abelian group, which fits into an exact sequence

$$0 \to H \to \mathrm{Aff}(N,H) \to \Lambda_{N/H} \to 0$$

with $\Lambda_{N/H}$ a lattice: the *character lattice* of N. The image in $\Lambda_{N/H}$ of a function $F \in \mathrm{Aff}(N,H)$ is its *differential* dF.

We can recover the H-rational points of N from its affine functions by the formula

$$N(H) = \mathrm{Hom}_H(\mathrm{Aff}(N,H),H),$$

where Hom_H denotes the set of group homomorphisms that commute with the structural maps from H. This set is a torsor for $\mathrm{Hom}(\Lambda_{N/H},H)$; in other words, $\Lambda^\vee_{N/H}\otimes H$ is the model space for $N(H)$. More generally, we may take points in any additive extension $H \subseteq H' \subseteq \mathbb{R}$ of H with the formula

$$N(H') = \mathrm{Hom}_H(\mathrm{Aff}(N,H),H').$$

If $N_1 \to N_2$ is an affine map of H-affine spaces, then we can form the exact sequence

$$0 \to \nu^\vee_{N_1/N_2} \to \mathrm{Aff}(N_2,H) \to \mathrm{Aff}(N_1,H) \to \Lambda_{N_1/N_2} \to 0$$

with $\Lambda^\vee_{N_1/N_2}\otimes H$ acting simply transitively on the fibres. If $\nu^\vee_{N_1/N_2} = \Lambda^\vee_{N_1/N_2} = 0$, then $N_1 \to N_2$ is a *lattice refinement* of affine spaces. An extension H' of H induces a natural refinement

$$N \to N \otimes H', \quad \mathrm{Aff}(N\otimes H',H') = \mathrm{Aff}(N,H)\oplus_H H'$$

of any H-affine space N (provided H is non-trivial). Of course, $N\otimes H'(H') = N(H')$.

We can form the quotient $N_2/\Lambda^\vee$ by a primitive distribution $\Lambda^\vee \subseteq \Lambda^\vee_{N_2/H}$ with affine functions the $\Lambda^\vee\otimes H$-invariants. They fit into a Cartesian square

$$\begin{tikzcd}
\mathrm{Aff}(N_2/\Lambda^\vee,H) \arrow[r] \arrow[d] & \ker[\Lambda_{N_2/H} \to \Lambda] \arrow[d] \\
\mathrm{Aff}(N_2,H) \arrow[r] & \Lambda_{N_2/H}
\end{tikzcd}$$

If the distribution $\Lambda = \Lambda_{N_1/H}$ comes from an affine subspace $N_2 \subseteq N_1$, we get a further Cartesian square

$$\begin{tikzcd}
\mathrm{Aff}(N_2/N_1) \arrow[r] \arrow[d] & \mathrm{Aff}(N_2,H) \arrow[d] \\
H \arrow[r] & \mathrm{Aff}(N_1,H)
\end{tikzcd}$$

whose left-hand vertical arrow splits the usual cotangent sequence. Thus $\mathrm{Aff}(N_2/N_1) \cong H \oplus \nu^\vee_{N_1/N_2}$ and $N_2/N_1(H)$ is in bijection with its model $\nu_{N_1/N_2}\otimes H$.

Any map of affine spaces can be factored into a surjection (purely transcendental submersion), a refinement (étale map), and a primitive embedding.

9.2 The cone of bounded functions

Let $\Delta \subseteq N$ be a polyhedron. The set $\mathrm{Aff}_\Delta(N,H)$ is a submonoid of $\mathrm{Aff}(N,H)$, with equality if Δ is of finite extent. When Δ is H-rational, $\mathrm{Aff}_\Delta(N,H)$ is a *cone over* H - a saturated monoid generated by H and finitely many additional elements. It fits into an exact sequence

$$0 \to H \to \mathrm{Aff}_\Delta(N,H) \to \Lambda_{\Delta/H} \to 0$$

whose right-hand term $\Lambda_{\Delta/H}$ is a polyhedral cone (finitely generated, saturated subgroup) inside $\Lambda_{N/H}$. Its polar $\Lambda^\diamond_{\Delta/H} \subseteq \Lambda^\vee_{N/H}$ is usually called the *recession cone* of Δ. It is the set of tangent vectors to rational rays contained within Δ.

The cone $\mathrm{Aff}_\Delta(N,H)$ is capable of separating points only up to the action of the *lineality space*

$$\Delta^{|} := \left(\frac{\mathrm{Aff}(N,H)}{\mathrm{Aff}_\Delta(N,H)}\right)^\vee \subseteq \Lambda^\vee_{N/H},$$

which is the largest linear subspace of $\Lambda^\diamond_{\Delta/H}$. Alternatively,

$$\mathrm{Hom}_H(\mathrm{Aff}_\Delta(N,H) \otimes \mathbb{Z}, H) = N/\Delta^{|}(H).$$

The lineality space is zero if and only if $\mathrm{Aff}_\Delta(N,H) \otimes \mathbb{Z} = \mathrm{Aff}(N,H)$, if and only if the recession cone (equivalently Δ) is *strongly convex*.

If $N_1 \to N_2$ maps $\Delta_1 \subseteq N_1$ into $\Delta_2 \subseteq N_2$, we once again have an exact sequence

$$0 \to v^\diamond_{\Delta_1/\Delta_2} \to \mathrm{Aff}_{\Delta_2}(N_2,H) \to \mathrm{Aff}_{\Delta_1}(N_1,H) \to \Lambda_{\Delta_1/\Delta_2} \to 0$$

of saturated monoids whose outer terms $v^\diamond_{\Delta_1/\Delta_2}, \Lambda_{\Delta_1/\Delta_2}$ are cones.

We may also form a quotient $\Delta_2/\Lambda^\vee$ of Δ_2 by a distribution $\Lambda^\vee \subseteq \Lambda^\vee_{N_2/H}$, whose H-points form the image of $\Delta_2(H)$ in $N_2/\Lambda^\vee(H)$, via the fibre square

$$\begin{tikzcd}
\mathrm{Aff}_{\Delta_2/\Lambda^\vee}(N_2/\Lambda^\vee,H) \arrow[r] \arrow[d] & \mathrm{Aff}(N_2/\Lambda^\vee,H) \arrow[d] \\
\mathrm{Aff}_{\Delta_2}(N_2,H) \arrow[r] & \mathrm{Aff}(N_2,H)
\end{tikzcd}$$

If the distribution comes from an embedding $N_1 \subseteq N_2$ of affine spaces, the affine functions on the quotient are $H \oplus v^\diamond_{\Delta_2/N_1}$.

Taking the quotient by the lineality space allows us to replace any polyhedron Δ with a strongly convex one $\Delta/\Delta^{|}$. All of the polyhedra in this document will be strongly convex.

9.3 The cone of non-positive functions

If $\mathrm{Aff}_\Delta(N,H)$ is generated by H and affine functions $F_1,\dots,F_k \in \mathrm{Aff}^+_\Delta(N,H)$, then Δ is an intersection of half-spaces

$$\Delta(H) = \bigcap_{i=1}^{k} F^{-1}H^\circ.$$

The recession cone $\Lambda^\diamond_{\Delta/H}$ is the largest submonoid of $\Lambda^\vee_{N/H}$ such that the action of $\Lambda^\diamond_{\Delta/H} \otimes H^\circ$ on $N(H)$ preserves $\Delta(H)$. It follows that

$$\Delta(H) = \mathrm{Hom}_{H^\circ}(\mathrm{Aff}^+_\Delta(N,H), H^\circ).$$

To put it another way, $\Delta(H)$ is the set of homomorphisms of *pairs*

$$\left(\mathrm{Aff}_\Delta(N,H);\mathrm{Aff}^+_\Delta(N,H)\right) \to (H;H^\circ)$$

that respect the H-structure.

The monoid $\mathrm{Aff}_\Delta(N,H)$ comes equipped with an H-indexed filtration

$$\mathrm{Aff}^+_\Delta(N,H) + \lambda \hookrightarrow \mathrm{Aff}_\Delta(N,H), \quad \lambda \in H$$

by $\mathrm{Aff}^+_\Delta(N,H)$-invariant subsets. It is automatically preserved by homomorphisms of pairs over H. Since every element of $\mathrm{Aff}^+_\Delta(N,H)$ is be bounded above by some constant, the filtration is *exhaustive*; since no function is bounded above by *every* constant, it is also *separated*.

9.4 Morphisms over $\mathbb{F}_1$ and the boundary at infinity

The set $\Delta(H)$ sits inside a natural 'compactification' $\Delta(H_\infty)$ in which certain strata, indexed by the faces of the recession cone, are added at infinity. Tthese strata are defined by allowing bounded functions to take the value $-\infty$. To describe this algebraically, we need to introduce *absorbing elements* to our monoids - in other words, move over $\mathbb{F}_1$.

Let Q be a monoid, $\mathbb{F}_1[z^Q]$ the associated $\mathbb{F}_1$-algebra. Let $I \trianglelefteq \mathbb{F}_1[z^Q]$ be an ideal. The quotient of $\mathbb{F}_1[z^Q]$ by I is, as a set, obtained by identifying the elements of $I \trianglelefteq \mathbb{F}_1[z^Q]$ with 0. In particular,

$$\mathbb{F}_1[z^Q] \setminus I \to (\mathbb{F}_1[z^Q]/I) \setminus 0$$

is *bijective*. If in particular $I = \mathfrak{p}$ is prime, then $\sigma_\mathfrak{p} := \mathbb{F}_1[z^Q] \setminus \mathfrak{p}$ is a submonoid of Q; in fact, it is a *face* in the sense that

$$X + Y \in \sigma_\mathfrak{p} \quad \Leftrightarrow \quad X, Y \in \sigma_\mathfrak{p}.$$

In particular, $\sigma_\mathfrak{p}$ contains every subgroup of Q. The quotient is uniquely identified with

$$\mathbb{F}_1[z^Q] \to \mathbb{F}_1[z^{\sigma_\mathfrak{p}}], \quad z^X \mapsto \begin{cases} z^X & \text{if } X \in \sigma_\mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$

Specialise now to the case of the $\mathbb{F}_1$-algebra $\mathcal{O}\{\Delta\}$ associated to $\mathrm{Aff}_\Delta(N,H)$. Since $H \subset \mathcal{O}\{\Delta\}$ is a group, it is contained in the complement of $\mathfrak{p}$. This complement is therefore the preimage of its image in the corecession cone $\Lambda_{\Delta/H}$, of which it is a face $\mathrm{asy}^\diamond_\mathfrak{p}$. The quotient is identified with $\mathcal{O}\{\Delta/\mathrm{asy}_\mathfrak{p}\}$, where $\mathrm{asy}_\mathfrak{p}$ is the polar cone to $\mathrm{asy}^\diamond_\mathfrak{p}$.

The cone $\mathrm{asy}_\mathfrak{p}$ could be called the *asymptotic cone* of Δ with limits in $\Delta/\mathrm{asy}_\mathfrak{p}$.

The sense in which $\Delta(H_\infty)$ is a 'compactification' of $\Delta(H)$ is as follows: let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence that escapes every bounded subset. Then if $\mathrm{asy}_\mathfrak{p}$ is the minimal asymptotic cone to x_n, the sequence can be said to *limit* into the infinite face $\Delta/\mathrm{asy}_\mathfrak{p}$ with affine functions $\mathcal{O}(\Delta)/\mathfrak{p}$; to be precise, its limit is the (eventually constant) image of the tail end of the sequence in $\Delta/\mathrm{asy}_\mathfrak{p}(H)$. This argument also shows that $\Delta(\mathbb{R}_\infty)$, equipped with the order topology coming from $\mathbb{R}$, is compact in the usual sense.

9.5 Faces

Let $v \in \Lambda_{\Delta/H}$. We can associate to v a *finite face* Δ_v of Δ along which any affine function integrating v is maximised. The function can be normalised to vanish along Δ_v; it is then in particular a member of $\mathrm{Aff}^+_\Delta(N,H)$. There is a pushout square

$$\begin{tikzcd}
v^\diamond_{\Delta_v/\Delta} \arrow[r] \arrow[d] & \mathrm{Aff}^+_\Delta(N,H) \arrow[d] \\
v^\diamond_{\Delta_v/\Delta} \otimes \mathbb{Z} \arrow[r] & \mathrm{Aff}^+_{\Delta_v}(N,H)
\end{tikzcd}$$

with $v^\diamond_{\Delta_v/\Delta}$ a cone in $\mathrm{Aff}^+_\Delta(N,H)$.

Looking at it another way, the finite faces of Δ are exactly the sub-polyhedra that are cut out by localisations of the $\mathbb{F}_1$-algebra $\mathcal{O}^+\{\Delta\}$ associated to $\mathrm{Aff}^+_\Delta(N,H)$. The character cone $\Lambda_{\Delta/H}$ is naturally subdivided by the corresponding conormal cones. By taking the cone over Δ, this recovers the familiar classification of open subsets of toric varieties discussed in I.6.2.

The usual convention is that faces in dimensions zero and one are called *vertices* and *edges*, and those in codimensions one and two are called *facets* and *ridges*, respectively.

Variant - Faces as closed strata at $t = 0$ Since $v^\diamond_{\Delta_v/\Delta}$ is a face of $\mathrm{Aff}^+_\Delta(N,H)$, its complement is a prime ideal in the corresponding $\mathbb{F}_1$-algebra $\mathcal{O}^+\{\Delta\}$. The quotient is dual to a closed subvariety of finite type over the residue field $\mathbb{F}_1$. The open subset defined by the localisation above is the minimal open subset of Δ that contains this closed subvariety.

9.6 Affine manifolds

An *H-affine manifold* is a manifold equipped with a reduction of structure group to the H-affine group $\mathrm{GL}_n(\mathbb{Z}) \ltimes H^n$. In other words, it is a manifold locally modelled on subsets of an H-affine space with affine transition maps.

Such a manifold comes equipped with a locally constant sheaf $\mathrm{Aff}(B,H)$ of *H-affine functions*. If H' is an extension of H, it is possible to make sense of the *H'-points* $B(H')$ of B as the set of points on which H-affine functions take values in H'. In particular, $B(\mathbb{R})$ is just the underlying topological space of B, equipped with the weak topology.

Let $\widetilde{B}$ be a universal cover of B. The global sections of $\mathrm{Aff}(\widetilde{B},H)$ are dual to an n-dimensional affine space N, and local charts patch together to yield a natural local affine diffeomorphism

$$\delta : \widetilde{B} \to N,$$

called the *developing map* of B. The obstruction to this map descending to B is the *monodromy representation*

$$\rho : \pi_1(B,p) \to \mathrm{Aut}(\mathrm{Aff}(N,H))$$

for any $p \in B(\mathbb{R})$. So this representation is trivial if and only if B is a covering space of an open subset of N. Of course, we can always make sense of the developing map locally on B.

Suppose that B is connected. Then the developing map is surjective if and only if all geodesics on B are parametrised by the entire real line, i.e. if B is *complete*. In this case, $\widetilde{B} \cong N$ and B is a $K(\pi,1)$.

The significance of the completeness hypothesis to analytic geometry is the conjecture of Markus:

Conjecture (Markus)**.** *An affine manifold is complete if and only if it has parallel volume.*

Certainly every complete orientable affine manifold has parallel volume; the converse is still open. Affine manifolds with parallel volume correspond to *Calabi-Yau* analytic spaces.

Closed, complete affine manifolds are classified by groups acting cocompactly and properly discontinuously by affine transformations on Euclidean space. Such subgroups of $\mathrm{Aff}(N)$ are also known as *affine crystallographic groups*. A classification is known in dimensions up to three; see [FG83].

It is also not difficult to formulate a notion of affine manifold *with corners at infinity*; I provide a sketch-definition here. An affine manifold with corners is a manifold with corners $(B,\partial B)$ with an affine structure on $B\setminus\partial B$ such that

- a geodesic limits into the boundary only when it is complete;
- each boundary stratum
$$\partial B_k \overset{i}{\hookrightarrow} B \overset{j}{\hookleftarrow} B\setminus\partial B_k$$
is an H-affine manifold with respect to the subsheaf $\mathrm{Aff}(\partial B_k,H)\subseteq i^*j_*\mathrm{Aff}(B,H)$ of locally bounded sections.

Affine functions on an affine manifold with corners are allowed to take the value $-\infty$. If the function is not everywhere $-\infty$, it can only take this value on the boundary.

10 Rigid analytic spaces

This presentation of analytic spaces is valid *verbatim* for ordinary rigid analytic spaces over $\mathbb{Z}$. For that reason, I have suppressed the subscript $\mathbb{F}_1$s for this and the next two sections.

The main examples having already been exposed in section 9, I have kept the discussion here mostly theoretical. I invite the reader also to keep in mind his favourite rigid analytic spaces over non-Archimedean fields as geometric motivation for this development.

10.1 Raynaud presentation

In §I.2.6 we defined a category ${}_Z\mathbf{FSch}$ of *marked formal schemes*, whose objects are pairs $(X^+;Z)$ consisting of a formal scheme X^+ and a finitely presented closed formal subscheme $Z\subseteq X^+$; morphisms of pairs are morphisms of formal schemes that, up to nilpotents, pull back target markings into source markings. The isomorphism class of a pair depends only on X^+ and the underlying reduced formal scheme of Z.

We will construct the rigid topos $\mathrm{Sh}\mathbf{Rig}$ so that it is universal with respect to the existence of a quasi-compact, quasi-separated geometric morphism

$$\mathrm{Sh}\mathbf{Rig} \xrightarrow{\zeta} \mathrm{Sh}_Z\mathbf{FSch}$$

such that $\zeta^* =: (-)\setminus\setminus Z$ inverts isomorphisms 'away from Z'. We will work on the principle that any such morphism can be dominated by a blow-up along a subscheme of Z, or more generally, a subscheme of X^+ whose reduction is contained in Z. Note that the definition of such morphisms is not local in Sh**Sch**, and so it will not be possible to consider formal schemes as embedded in that category.

The fact that ζ^* must be the pullback of a geometric morphism means that it is a localisation of $\mathrm{Sh}_Z\mathbf{FSch}$ only as a *stack* on itself, rather than as a plain (external) category. That is to say, it must preserve glueings, a.k.a. colimits. Moreover, the hypothesis that ζ be qcqs - by definition, meaning that ζ^* preserves compact objects - is essential if we are to have a reasonable theory of compactness for rigid analytic spaces.

Let us first define the category on compact objects - more precisely, on the site ${}_Z\mathbf{FSch}^{\mathrm{qcqs}}$ of qcqs marked formal schemes. Let us define W to be the class of admissible modifications (def. I.3.23). It is stable for composition. By lemma I.3.25, the saturation of W is stable for base change and descent.

We may now construct the category of *qcqs rigid spaces* as a localisation

$${}_Z\mathbf{FSch}^{\mathrm{qcqs}} \to {}_Z\mathbf{FSch}^{\mathrm{qcqs}}[W^{-1}] =: \mathbf{Rig}^{\mathrm{qcqs}}.$$

The following construction works for any left exact localisation. Let $\mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}}$ denote the category of ('admissible') pro-objects of $\mathbf{FSch}^{\mathrm{qcqs}}$ with transition maps in W. Since the saturation of W_{loc} is stable for base change, this category is closed under finite limits in the category of all pro-objects. The inclusion of ${}_Z\mathbf{FSch}^{\mathrm{qcqs}}$ as the constant pro-objects is therefore left exact.

We will let $\mathbf{Rig}^{\mathrm{qcqs}}$ be the full subcategory of $\mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}}$ spanned by the *colocal* objects, that is, the admissible pro-objects F for which $\mathrm{Hom}(F,X) \to \mathrm{Hom}(F,Y)$ is a bijection for any $X \to Y$ in W.

The inclusion of $\mathbf{Rig}^{\mathrm{qcqs}}$ has a right adjoint, which sends an admissible pro-object $\lim_i X_i$ to the cofiltered limit of all admissible blow-ups of the X_i.

$$\begin{tikzcd}
& \mathbf{Rig}^{\mathrm{qcqs}} \arrow[d, bend right] \\
{}_Z\mathbf{FSch}^{\mathrm{qcqs}} \arrow[r] & \mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}} \arrow[u, bend right]
\end{tikzcd}$$

The composite functor ${}_Z\mathbf{FSch}^{\mathrm{qcqs}} \to \mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}}$, which sends a marked formal scheme (X^+,Z) to the limit of blow-ups with centres in Z, is left exact.

10.1 Lemma. *Let $F : {}_Z\mathbf{FSch}^{\mathrm{qcqs}} \to \mathbf{C}$ be a functor inverting admissible blow-ups. It factors uniquely through $\mathbf{Rig}^{\mathrm{qcqs}}$.*

Proof. There is a unique extension $\mathrm{Pro}_Z F$ of F to $\mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}}$ that preserves limits of admissible blow-ups. This extension factors uniquely through the coreflective subcategory $\mathbf{Rig}^{\mathrm{qcqs}}$. Conversely, any factorisation of F through $\mathbf{Rig}^{\mathrm{qcqs}}$ restricts along the coreflector to a functor $\mathrm{Pro}_Z\mathbf{FSch}^{\mathrm{qcqs}} \to \mathbf{C}$, which sends limits of admissible blow-ups to constant diagrams and hence is uniquely identified with $\mathrm{Pro}_Z F$. □

10.2 Lemma (Local localisation)**.** *Let $\mathbf{S}$ be a topos, W a composable class of qcqs morphisms in $\mathbf{S}$. Let $\mathbf{S}^c$ be a finitely complete site for $\mathbf{S}$ whose objects are compact in $\mathbf{S}$ (so that $\mathbf{S}$ is coherent). Let $\zeta^* : \mathbf{S}^c \to \mathbf{S}^c[W^{-1}]$ be a left exact localisation.*[15]

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[15] To be precise, ζ^* is universal among functors to any category that invert the morphisms of W, and it is left exact. The existence of such a localisation depends only on the shape of W.

There exists a coherent topos $\mathbf{S}_W$ *and a qcqs geometric morphism* $\zeta : \mathbf{S}_W \to \mathbf{S}$ *whose pullback inverts morphisms represented by* W*, and is universal with respect to these properties. Moreover, there is a unique (up to unique isomorphism) commuting diagram*

$$\begin{tikzcd}
\mathbf{S}^c \arrow[r, "\zeta^*"] \arrow[d] & \mathbf{S}^c[W^{-1}] \arrow[d] \\
\mathbf{S} \arrow[r, "\zeta^*"] & \mathbf{S}_W
\end{tikzcd}$$

that makes $\mathbf{S}^c[W^{-1}]$ *into a finitely complete site of compact objects of* $\mathbf{S}_W$*.*

If $\mathbf{S}^c$ *is subcanonical and the morphisms in* W *are local, then* $\mathbf{S}^c[W^{-1}]$ *is subcanonical.*

Proof. This is probably well-known, but I include a proof here to save myself sifting through the literature. Put $\mathbf{S}_W$ the category of sheaves on $\mathbf{S}^c[W^{-1}]$ with respect to the coverage induced by ζ^*. It is initial among topoi fitting into a commuting square as above.

That the objects of $\mathbf{S}^c$ are compact in $\mathbf{S}$ is to say that the local isomorphisms in $\mathrm{PSh}\mathbf{S}^c$ are qcqs, that is, the base change of any such to a compact presheaf is compact. Since the morphisms of W are qcqs, so are the induced system of local isomorphisms in $\mathrm{PSh}(\mathbf{S}^c[W^{-1}])$. Hence the objects of $\mathbf{S}^c[W^{-1}]$ are compact in $\mathbf{S}_W$, which is therefore coherent. Moreover, since the compact objects of $\mathbf{S}$ (resp. $\mathbf{S}_W$) are the finite colimits of representable objects, and ζ^* preserves these, the latter is qcqs. ☐

Applying lemma 10.2, we obtain the *rigid topos* Sh**Rig**. The definition of (represented) *open immersion* in $\mathrm{Sh}_{\mathbb{Z}}$**FSch** induces a corresponding notion in Sh**Rig**. For a more explicit description of this notion, see the end of §10.2.

10.3 Definition (Rigid spaces as sheaves)**.** A *rigid analytic space* is a locally representable sheaf on $\mathbf{Rig}^{\mathrm{qcqs}}$. The category of rigid analytic spaces is denoted **Rig**.

The above arguments show that a rigid analytic space can locally be understood as a *pro-formal scheme*; for further remarks on this perspective, see §10.6.

In traditional rigid analytic geometry, one usually restricts attention to formal schemes which are punctured along the entirety of their reduction.

10.4 Definition (Purely analytic)**.** We call a rigid analytic space *purely analytic* if it admits no morphism from any formal scheme, that is, if $\zeta_* X = \emptyset$. A rigid space is purely analytic if and only if it locally has a formal model on which all algebraic subschemes are marked.

Base change Recall that in §I.2.4 we have defined a spatial geometric morphism

$$\pi : \mathrm{Sh}\mathbf{FSch}_{\mathbb{Z}} \to \mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}$$

so that $\mathbb{F}_1$-formal schemes may be base changed to ordinary formal schemes over $\mathbb{Z}$. By part *i)* of lemma I.3.25, up to saturation, this pullback functor also preserves W. It therefore descends to a spatial geometric morphism

$$\pi : \mathrm{Sh}\mathbf{Rig}_{\mathbb{Z}} \to \mathrm{Sh}\mathbf{Rig}_{\mathbb{F}_1}, \qquad \pi^* : \mathbf{Rig}_{\mathbb{F}_1} \to \mathbf{Rig}_{\mathbb{Z}}$$

so that rigid analytic spaces may be base changed from $\mathbb{F}_1$ to $\mathbb{Z}$.

A case of particular interest is the following. Let $\mathbb{F}_1((t))$ be the Laurent series field with the t-adic topology, K a non-Archimedean field (in the usual sense) equipped with a topological nilpotent t. There is a unique map $\mathbb{F}_1((t)) \to K$ that preserves t. This gives a family of base change operations

$$\mathbf{Rig}_{\mathbb{F}_1((t))} \to \mathbf{Rig}_K$$

parametrised by the open unit disc of K.

10.2 Models

The local existence of formal schemes modelling rigid spaces is fundamental to all aspects of their study.

A *model* for an object X of Sh**Rig** is an object of $\mathrm{Sh}_Z\mathbf{FSch}$ together with an isomorphism $X^+ \backslash Z \cong X$. If $X \in \mathbf{Rig}$, we will want to assume that models of X are formal schemes. Models of X form a category $\mathbf{Mdl}_X$, the fibre of ζ^* over X. We say that X has *enough models* if $\mathbf{Mdl}_X$ is cofiltered.

If X is qcqs, it has enough models. More generally:

10.5 Proposition. *A paracompact rigid space has enough models.*

The natural (counit) map $(X^+, Z) \to (X^+, \emptyset)$ in $\mathrm{Sh}_Z\mathbf{FSch}$ pulls back to a morphism

$$j : X \to X^+$$

in Sh**Rig**; conversely, this morphism characterises the model (X^+, Z), as we set Z equal to the union of all closed subschemes of X^+ whose pullback to X^+ is empty. We will usually denote the data of the formal model in terms of this morphism j.

10.6 Definitions (Representability conditions)**.** A morphism $f : X \to Y$ in Sh**Rig** is said to be *representable by formal schemes* if, for any qcqs rigid space Z and morphism $Z \to Y$, there exists a model of the base change $X \times_Y Z \to Z$ such that the square

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] & (X \times_Y Z)^+ \arrow[d, "f^+"] \\
Z \arrow[r] & Z^+
\end{tikzcd}$$

is Cartesian. In particular, f admits models locally on Y. As a morphism in $\mathrm{Sh}_Z\mathbf{FSch}$, f^+ is represented by a formal scheme in the sense of §10.1. If, more generally, X is a union of open subobjects that are representable by formal schemes over Y, we say that f is *locally representable by formal schemes*.

Let $\mathbf{P}$ be a property (resp. source-local property) of morphisms in Sh**FSch**. We say that a morphism in Sh**Rig** *has property* $^+\mathbf{P}$ if it is representable (resp. locally representable) by formal schemes, and moreover the (local) models f^+ can be chosen having property $\mathbf{P}$.

With $\mathbf{P}$ equal to one of the following properties, we suppress the superscript plus:

i) representable by schemes;

ii) open immersion;

iii) locally of finite type, presentation

iv) (formal) embedding;

v) (formally) finite, integral

vi) (formally) projective.

So, for example, a morphism of rigid spaces is projective if it admits a projective model. We specifically discuss the cases of separated, overconvergent, and proper morphisms in §11.

Suppose that X admits enough models. Any quasi-compact open immersion $U \hookrightarrow X$ admits a global model, and is thus given by the data of a model X^+ of X and a quasi-compact open subset U^+ of X^+. Note that even if X is $^+$Noetherian, so that X^+ is a Noetherian topological space, X itself typically has many non-quasi-compact open subsets.

10.3 Topologies on $\mathbb{F}_1$-algebras

The remarks of this section and the next are valid in principle for $\mathbb{F}_1$ and $\mathbb{Z}$, but to avoid technicalities I am only claiming precision for the $\mathbb{F}_1$ case. The study of the $\mathbb{Z}$ case is in any case well-established [Abb10, FK13].

As in §I.3.2, we will consider pairs $(A;A^+)$ consisting of an $\mathbb{F}_1$-algebra A^+ and an A^+-algebra A. Subsets of A invariant under the *ring of integers* A^+ will be called *discs*.

Except where otherwise specified, our pairs will satisfy the following assumption:

- A^+ is a saturated submonoid of A. *relatively normal*

In other words, if $f \in A$ and $f^n \in A^+$, then $f \in A^+$. By lemma I.3.29, any affine A-admissible modification of $\operatorname{Spec} A^+$ sits between A^+ and its saturation in A.

A *locally convex topology* on $(A;A^+)$ is a separated filtration by discs, which are declared *open*, which as in §I.2.3 we close under intersection and enlargement of discs. In particular, A^+ attains under the induced topology the structure of a linearly topologised ring. In fact, *in light of lemma I.2.6*, A is a pro-discrete A^+-module.[16]

As before, we will always want to assume

- the product of two open discs is open. *weak adicity*

This implies that A^+ is an adic ring in the sense of §I.2.3.

10.7 Aside. The reason I have added the prefix *weak* is that we might strengthen it to the condition

- A^+ - the empty product of discs - is open in A. *adicity*

which is traditional, and perhaps aesthetically pleasing as a complement to the previous axiom, but not strictly necessary for what follows. The geometric significance of the generalisation from adicity to 'weak' adicity is that we will be allowed to puncture formal schemes along closed, but not necessarily algebraic, formal subschemes. The only cost is that the completed localisations are not necessarily open homomorphisms; I do not know anything that would need such a hypothesis.

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[16] Emphasis added here to highlight the fact that this is *not* true over $\mathbb{Z}$.

A pair with these additional data fixed is called a *locally convex* $\mathbb{F}_1$*-algebra*. A ring homomorphism $A \to B$ preserving the ring of integers induces a pullback map on discs; this map is required to preserve the filtration (i.e. be continuous).

The structure sheaves in rigid geometry also have the following:

- A is a localisation of A^+. *Tate*

The nomenclature is an approximation to Huber's terminology 'Tate ring' [Hub96]. This condition is not preserved by limits, and so in sheaves of Tate rings the property holds only locally. We do at least have

10.8 Lemma. *Any product of Tate rings is Tate.*

Proof. Let $(A_i = A_i^+[S_i^{-1}]; A_i^+)$ be a family of Tate rings. Their product $\prod_i A_i$ is a localisation of $\prod_i A_i^+$ at all elements of the form $(1,\ldots,f,\ldots)$ with $f \in S_i$ in the ith position. ☐

10.9 Example. If H is a totally ordered group, the special $\mathbb{F}_1$-algebra $\mathbb{F}_1[z^H]$ could be called the 'equal characteristic one valuation field with value group H', and denoted accordingly $\mathbb{F}_1((t^{-H}))$. This notation presupposes that we consider it equipped with the 'valuation ring' $\mathbb{F}_1[[t^{-H}]]$ associated to H°, and the evident H-indexed filtration by $\mathbb{F}_1[[t^{-H}]]$-submodules. The sign convention is such that $t^{-\lambda}$ converges to 0 as $\lambda \to -\infty$ in H.

The point of the remarks of §9.3 is that the $\mathbb{F}_1$-algebra $\mathcal{O}\{\Delta\}$ associated to $\mathrm{Aff}_\Delta(N,H)$ is Hausdorff with respect to the t-adic topology, that is, it is a Banach $\mathbb{F}_1((t^H))$-algebra.

10.4 Affine presentation

The passage from the categorical to the algebraic picture of rigid analysis rests on a simple proposition:

10.10 Proposition. *Let X be a rigid space with model $j: X \to X^+$. Then $j_*\mathcal{O}_X^+$ is the integral closure of the image of $\mathcal{O}_{X^+}$ inside $j_*\mathcal{O}_X$.*

This is an immediate consequence of lemma I.3.29 for $\mathbb{F}_1$ and I.3.19 for $\mathbb{Z}$.

Let us denote by ${}_Z\mathbf{FSch}^{\mathrm{aff}}$ the full subcategory of ${}_Z\mathbf{FSch}$ whose objects (X^+, Z) are such that X^+ is affine and Z is a union of principal divisors. It generates ${}_Z\mathbf{FSch}^{\mathrm{div}}$, and since one can always admissibly blow up a marked formal scheme to get Z a union of Cartier divisors, it is also a site for the rigid topos Sh**Rig**.

Also, for this section, Alg will denote the category of (not necessarily relatively normal) Tate algebras (§10.3).[17]

Pushing forward the structure sheaf from $\mathbf{FSch}^{\mathrm{aff}}$, one obtains a sheaf $\mathcal{O}^+$ on ${}_Z\mathbf{FSch}^{\mathrm{aff}}$ of adic algebras. Let $S \subseteq \mathcal{O}^+$ denote the multiplicatively closed subsheaf of local sections $f \in \mathcal{O}^+$ such that $\mathcal{O}^+/f$ is supported on Z. Then $\mathcal{O} := \mathcal{O}^+[S^{-1}]$ is a sheaf of Tate algebras on ${}_Z\mathbf{FSch}^{\mathrm{aff}}$ with ring of integers $\mathcal{O}^+$.

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[17] One could alternatively take for ${}_Z\mathbf{FSch}^{\mathrm{aff}}$ the category on which Z is formed of Cartier divisors. This would correspond on the algebraic side to the assumption that $A^+ \subseteq A$.

Let $(A;A^+)$ be a Tate algebra. The set of elements of A^+ that become invertible in A define a collection of Cartier divisors on $\operatorname{Spec} A^+$, and hence an object of ${}_Z\mathbf{FSch}^{\mathrm{aff}}$. This construction is contravariantly functorial

$$\operatorname{Spec} : \mathrm{Alg} \to {}_Z\mathbf{FSch}^{\mathrm{aff}}$$

and it is a spectrum functor in the sense of ringed toposes, that is, it is *left adjoint* to the structure sheaf

$$\mathcal{O} : {}_Z\mathbf{FSch}^{\mathrm{aff}} \to \mathrm{Alg}.$$

In fact, the two functors are inverse equivalences of categories.

10.11 Definition. A rigid analytic space is said to be *affine* if it admits a model in ${}_Z\mathbf{FSch}^{\mathrm{aff}}$. The full subcategory of **Rig** whose objects are affine is denoted $\mathbf{Rig}^{\mathrm{aff}}$.

The composite of the Spec functor with the localisation

$${}_Z\mathbf{FSch}^{\mathrm{aff}} \to \mathbf{Rig}^{\mathrm{aff}}$$

remains left adjoint to $\mathcal{O}$, but it is no longer an equivalence: the counit $A \to \mathcal{O}^+(\operatorname{Spec} A)$ need not be an isomorphism. To correct this, we must invert the morphisms in Alg dual to affine admissible blow-ups.

We have identified these homomorphisms: by proposition 10.10, they are precisely the *Z-admissible integral extensions*. In particular, $\mathcal{O}(\operatorname{Spec} A)$ is exactly the *relative normalisation* of A. This is why we usually - indeed, henceforth - make the assumption that Tate rings are relatively normal.

10.12 Proposition. Spec *and* $\mathcal{O}$ *form inverse anti-equivalences between* $\mathbf{Rig}^{\mathrm{aff}}$ *and the category of relatively normal Tate rings.*

Dually, this result reflects the fact that the localisation on affine objects has a fully faithful right adjoint $\mathbf{Rig}^{\mathrm{aff}} \hookrightarrow {}_Z\mathbf{FSch}^{\mathrm{aff}}$. In other words,

10.13 Corollary. *An affine rigid space X has a* unique *affine, relatively normal model given by the spectrum of $\mathcal{O}^+(X)$.*

We conclude by describing the presentation of Sh**Rig** by means of $\mathbf{Rig}^{\mathrm{aff}}$. An open immersion in $\mathbf{Rig}^{\mathrm{aff}}$ with target $\operatorname{Spec} A$ is given by the base change to A of a principal affine subset of an admissible modification of $\operatorname{Spec} A^+$, which up to relative normalisation is just an admissible blow-up. It has co-ordinate algebra of the form

$$A \to A\{T/s\} = \left(A \otimes_{A^+} A^+\{T/s\}; A^+\{T/s\}\right)$$

with T of finite type, $s \in T$ and $TA = A$. Since T/s is an open disc, $A\{T/s\}$ is a Banach module over A.

Conversely, any *completed localisation* of A - that is, algebra of the form $A\{T/s\}$ with T finitely generated - comes from an affine open immersion. In other words, a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ in $\mathbf{Rig}^{\mathrm{aff}} \subset \mathrm{Alg}^{\mathrm{op}}$ is an open immersion if and only if B is the relative normalisation of a completed localisation of A.

The following are equivalent for a finite family of completed localisations $A \to A\{T_i/s_i\}$:

i) $\operatorname{Spec} A = \bigcup_i \operatorname{Spec} A\{T_i/s_i\}$;

ii) there exist admissible blow-ups $X^+ \to \operatorname{Spec} A^+$ and $U_i \to \operatorname{Spec} A^+\{T_i/s_i\}$ such that $U_i^+ \hookrightarrow X^+$ and $X^+ = \bigcup_i U_i^+$;

iii) $A \to \prod_i A\{T_i/s_i\}$ is a universally effective monomorphism in $\mathrm{Alg}_{\mathbb{F}_1}$.

These criteria can be used to present the rigid topos and **Rig** in terms of topological algebra.

10.5 Summary of categories

I introduced a lot of categories in this section, so let me draw most of the main players out here as what I hope to be a handy reference utility.

$$\begin{tikzcd}
\mathbf{Sch}^{\mathrm{aff}}_{\mathbb{F}_1} \arrow[r] \arrow[dd] & \mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1} \arrow[dd] & {}_Z\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1} \arrow[l] \arrow[r] \arrow[d] & \mathbf{Rig}^{\mathrm{aff}}_{\mathbb{F}_1} \arrow[d] \\
 & & {}_Z\mathbf{FSch}^{\mathrm{qcqs}}_{\mathbb{F}_1} \arrow[r] \arrow[d] & \mathbf{Rig}^{\mathrm{qcqs}}_{\mathbb{F}_1} \arrow[d] \\
\mathbf{Sch}_{\mathbb{F}_1} \arrow[r] \arrow[dr] & \mathbf{FSch}_{\mathbb{F}_1} \arrow[d] & {}_Z\mathbf{FSch}_{\mathbb{F}_1} \arrow[l] \arrow[r] \arrow[d] & \mathbf{Rig}_{\mathbb{F}_1} \arrow[d] \\
 & \mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1} & \mathrm{Sh}{}_Z\mathbf{FSch}_{\mathbb{F}_1} \arrow[l] \arrow[r, "\zeta^*"] & \mathrm{Sh}\mathbf{Rig}_{\mathbb{F}_1}
\end{tikzcd}$$

I remind the reader of the following:

- The categories in the top row are opposite to the category of $\mathbb{F}_1$-algebras, pro-discrete $\mathbb{F}_1$-algebras, Tate $\mathbb{F}_1$-algebras, and relatively normal Tate $\mathbb{F}_1$-algebras, respectively.
- The topoi in the bottom row are each generated by any category vertically above, *except* for ${}_Z\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1} \subset \mathrm{Sh}{}_Z\mathbf{FSch}_{\mathbb{F}_1}$ (§10.4). The arrows in the bottom row are the pullback functors for geometric morphisms.
- $\mathrm{Sh}\mathbf{FSch}_{\mathbb{F}_1}$ is actually a presheaf category on $\mathbf{FSch}^{\mathrm{aff}}_{\mathbb{F}_1}$.
- The horizontal arrow in the top-right and the one immediately below are left exact categorical localisations.

We also have used intermediate marking categories

$$ {}_Z\mathbf{FSch}^{\nu}_{\mathbb{F}_1} \subset {}_Z\mathbf{FSch}^{\mathrm{inv}}_{\mathbb{F}_1} \subset {}_Z\mathbf{FSch}^{\mathrm{div}}_{\mathbb{F}_1} \subset {}_Z\mathbf{FSch}_{\mathbb{F}_1} $$

whose markings are relatively normal Cartier, Cartier, and divisorial, respectively.

10.6 Riemann-Zariski space

We have constructed the category of qcqs rigid analytic spaces as a subcategory of the category of pro-formal schemes; by definition, the rigid space associated to a marked rigid space (X^+, Z) is a formal limit

$$X = \lim \widetilde{X}^+$$

over admissible blow-ups $\widetilde{X}^+$ of X^+. By understanding this limit instead in the category of locally linearly-topologised-monoidal topological spaces, we can define a space $\mathrm{RZ}(X)$, the *Riemann-Zariski space* of X (or (X^+, Z)). The structure of the terms $\widetilde{X}^+$ as objects of ${}_Z\mathbf{FSch}$ equips the limit with a sheaf $(\mathscr{O}_X; \mathscr{O}_X^+)$ of Tate $\mathbb{F}_1$-algebras (though recall that the Tate condition does not persist on the spaces of sections over large open sets of $\mathrm{RZ}(X)$).

The topology on $\mathrm{RZ}(X)$ is that of the limit; it is therefore generated by the quasi-compact open sets of models X^+, with two such sets being equal if and only if they agree on a coinitial family of models. It is equal to the lattice of open subobjects of X in Sh**Rig**. In particular, $\mathrm{RZ}(X)$ is a quasi-compact and quasi-separated topological space.

It follows that an open immersion $U \hookrightarrow X$ of (qcqs) rigid spaces gives rise to an open immersion of their Riemann-Zariski space, and hence that this construction can be globalised to a functor

$$\mathbf{Rig}_{\mathbb{F}_1} \to \mathbf{Top}$$

into the category of topological spaces. As usual, this functor can be upgraded to take values in the category of topological spaces equipped with a sheaf of Tate $\mathbb{F}_1$-algebras. We will usually not distinguish between a rigid analytic space and its Riemann-Zariski space.

One can give a point-set-topological definition of rigid spaces in this manner.

10.14 Definition (Rigid spaces as Tate-ringed topological spaces)**.** Let (X^+, Z) be an object of ${}_Z\mathbf{FSch}_{\mathbb{F}_1}$, that is, a formal scheme $X^+ \in \mathbf{FSch}_{\mathbb{F}_1}$ equipped with a closed subset Z. Define a Tate-ringed topological space $\mathrm{RZ}(X^+, Z)$ by the preceding formula, limiting over all blow-ups of X^+ along Z.

A *rigid analytic space* is a Tate-ringed topological space X locally modelled by $\mathrm{RZ}(X^+, Z)$ for some $(X^+, Z) \in {}_Z\mathbf{FSch}_{\mathbb{F}_1}^{\mathrm{qcqs}}$. Note that on sufficiently small open sets $U \subseteq X$, one always has a canonical reprentative $\mathrm{Spec}\,\mathscr{O}(X)$.

It is quite likely that the underlying set of a Riemann-Zariski space can be described in terms of valuations; for a special case, see the appendix A.

11 Overconvergence in rigid analytic geometry

In §4, we defined notions of overconvergence via the existence of solutions to certain *extension problems*. In applying this to rigid geometry, we should keep in mind that the primary desired property for proper morphisms of rigid spaces is that they should admit proper models; this is proposition 11.9.

An alternative approach would be to simply define proper morphisms to be those admitting a proper model. This would remove some of the difficulties encountered below. At least for paracompact morphisms, it would even be possible to extend this approach to defining overconvergence; cf. 11.10.

11.1 Rigid analytic closure operator

To avoid constantly having to pick models, it will be convenient to be able to pass to the *formally embedded closure* at the level of rigid spaces. This is a simple special case of the construction in §I.4.1 of the overconvergent germ with respect to $\mathbb{P}$ the class of formal embeddings. This class evidently satisfies (P1-4), and (SC) is trivial since the property of being a formal embedding is local on the target.

Let $U \hookrightarrow V$ be an open immersion in **Rig**, and suppose for now that V is paracompact and, in the $\mathbb{F}_1$ case, that U/V is modelled by an *affine* open immersion $U^+ \hookrightarrow V^+$ of formal schemes. Then

$$\mathrm{cl}(U^+/V^+)\backslash\backslash Z \hookrightarrow V$$

is formally embedded. If $\tilde{V}^+ \to V^+$ is a Z-admissible modification with base change $\tilde{U}^+$ to U^+, then

$$\mathrm{cl}(\tilde{U}^+/\tilde{V}^+) \hookrightarrow \mathrm{cl}(U^+/V^+) \times_{V^+} \tilde{V}^+$$

is an affine formal embedding.

By ranging over models $U^+ \subseteq V^+$ of U/V, we may therefore define a *rigid analytic closure* as a pro-object

$$\mathrm{cl}(U/V) = \lim_{U^+/V^+} \mathrm{cl}(U^+/V^+)\backslash\backslash Z \in \mathrm{Pro}(\mathbf{Rig}_V)$$

in the category of rigid spaces over V.

11.1 Aside. Since the transition maps are modelled by affine morphisms of formal schemes, this pro-object can actually be realised as a rigid analytic space over V, though it is not formally embedded.

In other words, $\mathrm{cl}(U/V)$ is actually a sheaf of pro-objects on $\mathscr{U}^{\mathrm{qcqs}}_{/V}$, which we confuse with its global sections. We can use this fact to extend the definition to arbitrary V.

For general V, we cannot guarantee that $\mathrm{cl}(U/V)$ is contained in any formally embedded subspace. Despite this, the statement $\mathrm{cl}(U/V) \subseteq Z$ is still well-formed for any immersed $Z \hookrightarrow X$; in particular, for open subsets.

11.2 Extension problems for rigid spaces

We define $f\mathbb{P}$, $fi/\mathbb{P}$ in the topos of marked formal schemes to be the classes of morphisms represented by formally projective, resp. formally integral/projective, morphisms. The forgetful functor

$$\mathrm{Sh}_Z\mathbf{FSch} \hookrightarrow \mathrm{Sh}\mathbf{FSch}$$

both preserves and detects overconvergence. It follows that both its adjoints do as well. The proof being more a problem of notation than anything else, I omit it.

The essential image of the classes $f\mathbb{P}$, $fi/\mathbb{P}$, in the rigid topos are also called formally projective (resp. integral/projective) morphisms, cf. def. 10.6. Since admissible modifications are formally integral/projective, the existence of one $fi/\mathbb{P}$ model for a morphism implies that of a coinitial family (locally on the target).

11.2 Lemma. *The localisation*

$$\mathrm{Sh}_Z\mathbf{FSch} \to \mathrm{Sh}\mathbf{Rig}.$$

preserves overconvergence.

Proof. Let V be any qcqs rigid space, $U \hookrightarrow V$ an open immersion. Then

$$\mathrm{Sur}_{U/V} \to \mathrm{Sur}_{U^+/V^+} \setminus\setminus Z$$

is, as a pro-object of Sh**Rig**, a cofiltered limit over models U^+/V^+ of U/V. It follows that for a marked formal scheme (S^+, Z) and structural map $V \to S := S^+ \setminus\setminus Z$,

$$\operatorname*{colim}_{V^+} \mathrm{Hom}_{S^+}(\mathrm{Sur}_{U^+/V^+}, X^+) \tilde{\to} \mathrm{Hom}_S(\mathrm{Sur}_{U/V}, X^+ \setminus\setminus Z)$$

for any marked formal scheme X^+/S^+. We have shown criterion *iv)* of lemma I.4.3. □

For the rigid analytic puncturing to *detect* overconvergence, we will certainly need at least to restrict to the category ${}_Z\mathbf{FSch}^{\mathrm{inv}}$ of formal schemes marked along Cartier divisors. Under this restriction, we may try to apply criterion *iii)* of lemma I.4.3 by showing that the square

$$\begin{tikzcd}
\mathrm{Hom}_{S^+}(\mathrm{Sur}_{U^+/V^+}, -) \arrow[r] \arrow[d] & \mathrm{Hom}_S(\mathrm{Sur}_{U/V}, -) \arrow[d] \\
\mathrm{Hom}_{S^+}(U^+, -) \arrow[r] & \mathrm{Hom}_S(U, -)
\end{tikzcd}$$

is Cartesian for any $U^+/V^+/S^+$.

The problem is that the existence of a lift $\mathrm{Sur}_{U/V} \to X$ of a morphism $U \to X$ only, *a priori*, guarantees that we can find a morphism into X^+ from a modification $\tilde{V}^+ \to V^+$ restricting to an *admissible* modification $\tilde{U}^+$ of U^+:

$$\begin{tikzcd}
\widetilde{U}^+ \arrow[r] \arrow[d, "Z\text{−adm.}"'] & \widetilde{V}^+ \arrow[d, "f\mathbb{P}"] \arrow[ddl, dashed] \\
U^+ \arrow[r] \arrow[d] & V^+ \arrow[d] \\
X^+ \arrow[r, "f"] & S^+
\end{tikzcd}$$

Typically $\widetilde{V}^+$ will not admit a section over U^+.

We would like, nonetheless, for these data to give rise to a unique morphism $\mathrm{Sur}_{U^+/V^+} \to X^+$ over S^+. In other words, we want that $\mathrm{Sur}_{\widetilde{U}^+/\widetilde{V}^+}$ and U^+ form a *canonical covering* of Sur_{U^+/V^+} in the category of algebraic spaces.

11.3 Lemma. *Let $\widetilde{V} \to V$ be surjective. The square*

$$\begin{tikzcd}
\widetilde{U} \arrow[r] \arrow[d] & \mathrm{Sur}_{\widetilde{U}/\widetilde{V}} \arrow[d] \\
U \arrow[r] & \mathrm{Sur}_{U/V}
\end{tikzcd}$$

is a pushout in the category of formal algebraic spaces separated and locally of finite type over S.

Proof over $\mathbb{Z}$. Suppose we are given a square

$$\begin{tikzcd}
\widetilde{U} \arrow[r] \arrow[d] & \widetilde{V} \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

and, without loss of generality, that U is dense in V. We may assume that all players are schemes; the general result follows from passing to an inductive colimit. In particular, $\widetilde{V} \to V$ is projective. We suppress S from the notation.

We may obtain a morphism into X from the closure $\tilde{V}$ of U in $X \times V$. Since X is separated and locally of finite type, $\tilde{V}$ is separated and of finite type over V. It will be enough to show that $\tilde{V} \to V$ is proper, and hence that $\mathrm{Sur}_{U/V} \to \tilde{V}$.

Because $\widetilde{U}$ is dense in $\widetilde{V}$, the section $\widetilde{V} \to X \times V$ factors through $\tilde{V}$. By [Gro60, **II**.5.4.3.*i)*], $\widetilde{V} \to \tilde{V}$ is proper. In particular, it is closed; being surjective over a dense open subset, it is therefore surjective. Finally, by [Gro60, **II**.5.4.3.*ii)*], $\tilde{V} \to V$ is proper. ☐

Essentially the same argument would carry through in the $\mathbb{F}_1$ case if we had the analogue of [Gro60, **II**.5.4.3.*ii)*] (cf. lemma I.4.51) for proper morphisms over $\mathbb{F}_1$ and if we knew that torsion-free modifications were 'strongly surjective' in the sense of *loc. cit.*.

Unfortunately, we don't have either of these things, and so we are reduced to pursuing a more ad hoc approach using the combinatorial methods of §I.7.4:

Proof over $\mathbb{F}_1$. Let us fix a square

$$\begin{tikzcd}
\widetilde{U} \arrow[r] \arrow[d] & \widetilde{V} \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

and assume that all players are schemes and that U/V is affine and dense.

Toric case. Suppose that X/S lives in $\mathbf{Sch}_{\mathbb{F}_1}^{\mathrm{qi/nb}}$ and that S is Noetherian. We may as well assume that X is connected; it is then represented by a fan $\Sigma_X \subset N_X$. The morphism $\sigma_U \to \Sigma_X$ induces a map $\sigma_V(\mathbb{R}) \to N_X(\mathbb{R})$ whose image, because $\sigma_{\widetilde{V}} \to \sigma_V$ is bijective, is in the support of Σ_X. We therefore obtain a morphism $\tilde{\sigma}_V \to \Sigma_X$ from a subdivision $\tilde{\sigma}_V$ of σ_V not touching σ_U.

Noetherian case. Suppose that S is Noetherian. Then V and $\widetilde{V}$ are Noetherian and hence can be decomposed into finitely many quasi-integral closed subschemes $V_i, \widetilde{V}_i$ which, for simplicity, we index by the same set.

By replacing X with the closure X_i of the image of U_i, we reduce to the toric case. We therefore have unique extensions $\mathrm{Sur}_{U_i/V_i} \to X$. By proposition I.5.7, these extensions glue to a unique extension $\mathrm{Sur}_{U/V} \to X$.

General case. Let us replace X with a quasi-compact open subset through which $\widetilde{V} \to X$ factors. All objects are now of finite type over S, and so there exists a Noetherian formal scheme S_0, a diagram

$$\begin{tikzcd}
\widetilde{U}_0 \arrow[r] \arrow[d, "Z\text{-adm.}"'] & \widetilde{V}_0 \arrow[d, "\mathbb{P}"] \arrow[ddl, dashed] \\
U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\
X_0 \arrow[r, "f"] & S_0
\end{tikzcd}$$

over S_0 and an embedding $X \hookrightarrow X_0 \times_{S_0} S$. By the Noetherian case, there is a unique map $\mathrm{Sur}_{U_0/V_0} \to X_0$ descending from $\mathrm{Sur}_{\widetilde{U}_0/\widetilde{V}_0}$, and hence an extension

$$\mathrm{Sur}_{U/V} \to \mathrm{Sur}_{U_0/V_0} \times_{S_0} S \to X_0 \times_{S_0} S.$$

Since embeddings are proper (proposition I.4.42), the extension actually factors through the embedded formal subscheme X. □

11.4 Aside (Descent along blow-downs). In certain cases, it may even be possible to construct a pushout of $\widetilde{U}^+ \to \widetilde{V}^+$ along the blow-down $\widetilde{U}^+ \to U^+$; for instance, this is what is happening in proposition I.5.7.

Here is a sketch of an algorithmic construction in the *toric* case, that is, when the square lives in $\mathbf{FSch}^{\mathrm{n/nb}}_{\mathbb{F}_1}$. We may assume that all players are Noetherian. Then $\Sigma_{\tilde{V}^+} \to \Sigma_{V^+}$ is a subdivision of a neighbourhood of Σ_{U^+}. Let us consider these as fans immersed in N_{U^+}. The objective is to 'coarsen' $\Sigma_{\tilde{V}^+}$ in a minimal way such that it no longer subdivides Σ_{U^+}.

Let us begin by deleting the cones of $\Sigma_{\tilde{V}^+}$ that intersect Σ_{U^+} in some set that is not a face. The resulting collection of cones may fail to be strongly convex, so we continue by deleting cones where this failure occurs. Since $\Sigma_{\tilde{V}^+}$ has finitely many cones, this procedure terminates after a finite number of steps, and its termination implies that we are left with a punctured fan. This object is the blow-down.

11.3 Separation

11.5 Definition. A morphism of rigid spaces is said to be *locally separated* if it is locally $f\mathbb{P}$-separated (def. 4.3). It is *separated* if it is quasi-separated and locally separated.

By lemma 11.2, if $X^+ \to S^+$ is a (locally) separated morphism of formal schemes, then $X^+ \backslash\backslash Z \to S^+ \backslash\backslash Z$ is (locally) separated for any marking Z of X^+ and S^+. Conversely:

11.6 Lemma. *Let $X^+ \to S^+$ be a morphism of formal schemes with Cartier marking.*

If $X^+ \backslash\backslash Z \to S^+ \backslash\backslash Z$ is locally separated, then X^+/S^+ is locally separated.

Proof. Let

$$\begin{tikzcd}
U^+ \arrow[r] \arrow[d] & V^+ \arrow[dl, shift left] \arrow[dl, shift right] \\
X^+ &
\end{tikzcd}$$

be an extension problem with two solutions. Since X is locally separated, the two morphisms become equal after a Z-admissible modification $\tilde{V}^+$ of V^+.

As Z is invertible, $\tilde{V}^+ \to V^+$ is surjective, and so $V^+ \rightrightarrows X^+$ both have the same underlying map of sets. Assuming V^+ affine, we may therefore replace X with an affine open subset through which both solutions factor. But $U^+ \to V^+$ is an epimorphism in the category of affine schemes, so the two arrows are equal. □

11.7 Proposition. *Let $X \to S$ be a quasi-separated morphism of analytic spaces. The following are equivalent:*

i) X/S is separated;

ii) every quasi-compact open subset of X is separated over S;

iii) locally on S, every quasi-compact open subset of X admits a separated model;

iv) every model with Cartier marking of every open subset of X is separated;

v) the diagonal of X/S is an embedding.

Proof. The equivalence *i)⇔ii)* is elementary. We have just seen, through lemmas 11.2 and 11.6, the equivalence with *iii)* and *iv)*. Finally, *v)* is a consequence of corollary I.4.43 applied to local models of the diagonal, which is qcqs by hypothesis. □

11.4 Overconvergence and propriety

11.8 Definition. A morphism $X \to S$ is said to be *overconvergent*, resp. *proper*, if it is $f\mathbb{P}$-overconvergent, resp. $f\mathbb{P}$-proper (def. I.4.3) and locally of finite type.

By lemma 11.2, if $X \to S$ is an overconvergent (resp. proper) morphism of formal schemes, then $X \setminus\setminus Z \to S \setminus\setminus Z$ is overconvergent (resp. proper) for any marking of X and S.

The converse is a little more difficult:

11.9 Proposition. *Let $f : X \to S$ be a morphism of analytic spaces. The following are equivalent:*

i) f is proper;

ii) locally on S, f admits a proper model;

iii) a model with Cartier marking of a base change of f is proper.

Proof. A proper morphism is qcqs and so certainly admits a model locally on the base. The trick is to show that this model is proper whenever the marking is Cartier. By proposition 11.7, it is at least separated. But then the existence of solutions to extension problems follows from lemma 11.3. □

By choosing models, it follows immediately from proposition I.4.46:

11.10 Corollary. *If f is paracompact, then the following are equivalent:*

i) f is overconvergent;

ii) every proper X-space qcqs over S is proper over S;

iii) every formally embedded subspace of X qcqs over S is proper over S.

11.5 Overconvergence après Deligne

In [Del92], Deligne defines a sheaf F on the small topos $\mathrm{Sh}(X)$ of a quasi-separated rigid analytic space X over $\mathbb{Z}$ to be overconvergent if and only if, for all qcqs open $U \subseteq X$,

$$\operatorname*{colim}_{U \subseteq \mathrm{cl}(U/X) \subseteq V} F(V) \to F(U)$$

is an isomorphism.[18] The term on the left does not depend on whether we interpret $\mathrm{cl}(U/X)$ in terms of point set topology or as a pro-object as in §11.1. As such, the definition is equally well-formed over $\mathbb{F}_1$.

By definition of the pullback from $\mathrm{Sh}(X)$ to $\mathrm{Sh}\mathbf{Rig}$, this term is simply the value of F on the pro-object $\mathrm{cl}(U/X)$. Thus the definition equivalently says that every diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \mathrm{cl}(U/X) \arrow[r] \arrow[dl, dashed] & X \arrow[d, equal] \\
F \arrow[rr] & & X
\end{tikzcd}$$

has a unique extension $\mathrm{cl}(U/X) \to F$.

This is a particular case of an extension problem with $\mathbb{P}$ the class of formal closed embeddings. Since a morphism $U \to F$ from an arbitrary object of $\mathrm{Sh}\mathbf{Rig}_X$ locally factors through an open subset of X, it follows that we have unique solutions for any extension problem U/V. Thus we arrive at the following, equivalent for quasi-separated X, formulation of Deligne's definition:

11.11 Definition. A small sheaf on a rigid analytic space X (over $\mathbb{F}_1$ or $\mathbb{Z}$) is *Deligne overconvergent* if it is $\mathbb{P}$-overconvergent as an object of $\mathrm{Sh}\mathbf{Rig}_X$ with $\mathbb{P}$ the class of formal embeddings.

Deligne overconvergence implies overconvergence in the sense of definition 11.8. For the converse statement, we have the following (compare lemma I.4.41):

11.12 Lemma. *Let $U \hookrightarrow V$ a quasi-compact open immersion of X-spaces. The natural map*

$$\mathrm{Hom}_X(\mathrm{cl}(U/V), -) \xrightarrow{\sim} \mathrm{Hom}_X(\mathrm{Sur}_{U/V}, -)$$

is an isomorphism of functors on $\mathrm{Sh}(X)$*.*

Proof. Let $F \in \mathrm{Sh}(X)$ and $\mathrm{Sur}_{U/V} \to F$. By definition,

$$F(\mathrm{Sur}_{U/V}) \cong \operatorname*{colim}_{\tilde{V} \to V} \operatorname*{colim}_{\tilde{V} \to W \subseteq X} F(W)$$

locally on V; so any $f \in F(\mathrm{Sur}_{U/V})$ is represented by a section $W \to F$ such that $\tilde{V} \to W$ for some formally projective modification $\tilde{V} \to V$. We will show that this implies $V \to W$.

Without loss of generality, suppose V is affine, and that we have models

$$\begin{tikzcd}
& \tilde{V}^+ \arrow[d] \\
U^+ \arrow[ur] \arrow[r] & V^+
\end{tikzcd}$$

---

[18] Actually, Deligne works only with rigid analytic spaces admitting a Noetherian formal model, but the definition works without modification in an arbitrary quasi-separated geometry.

such that $U^+ \subseteq V^+$ is dense. By compactness we may also assume that W is qcqs, and thus that $W \times_X V \hookrightarrow V$ has a model $\tilde{V}^+ \to W^+ \subseteq V^+$.

But $\tilde{V}^+ \to V^+$ is surjective, and so $W^+ = V^+$. □

11.13 Proposition. *A sheaf on the small site of a rigid analytic space is overconvergent if and only if it is Deligne overconvergent.*

11.14 Corollary. *Let $U \subseteq X$ be an open subset of a quasi-separated analytic space X. The following are equivalent:*

i) *$U \subseteq X$ is overconvergent;*

ii) *for any $W \subseteq U$ affine over X, U contains* $\mathrm{cl}(W/X)$*;*

iii) *for any $W \subseteq U$ quasi-compact over X, U contains* $\mathrm{cl}(W/X)$.

11.6 Overconvergent site

The idea of recovering Berkovich's Hausdorff topology as a 'coarsening' of the topology of rigid analytic spaces is apparently also due to Deligne [Del92]. I learned about it from [Hub96, §8], where it appeared under the name 'partially proper' topology.

The generalities in this paragraph make sense for any spatial geometric context with a notion of overconvergence generated by a class of morphisms $\mathbb{P}$ as in §I.4; in particular, we will apply it to the topos of *collages* below in §12. For simplicity and concreteness, we restrict attention here to the motivating case of rigid analytic spaces.

Large site Let X be a rigid analytic space over $\mathbb{F}_1$ or $\mathbb{Z}$, and let $\mathrm{Sh}\mathbf{Rig}_X^{\mathrm{sur}}$ denote the *overconvergent topos* of X, that is, the category of overconvergent sheaves over X. By part *v)* of proposition I.4.6, every morphism in $\mathrm{Sh}\mathbf{Rig}_X^{\mathrm{sur}}$ is overconvergent. The category $\mathbf{Rig}_X^{\mathrm{sur}}$ of rigid analytic spaces overconvergent over X is a full subcategory.

11.15 Lemma. $\mathbf{Rig}_X^{\mathrm{sur}}$ *is a spatial site (def. I.1.1) for* $\mathrm{Sh}\mathbf{Rig}_X^{\mathrm{sur}}$.

Proof. Since overconvergence is local on X, we may assume that X is quasi-separated. We first show that the overconvergent topos is generated by analytic spaces formally *projective* over X. Since every object $F \in \mathrm{Sh}\mathbf{Rig}_X^{\mathrm{sur}}$ is a colimit in $\mathrm{Sh}\mathbf{Rig}_X$ of affine rigid analytic spaces of finite type over X, it will be enough to show that every morphism $U \to F$ from such a space U factors through some object formally projective over X.

Since U is of finite type over X, it may be embedded in some projective bundle $\mathbb{P}(\mathscr{E})$. Overconvergence means that after replacing the latter with a formally projective modification, $\mathbb{P}(\mathscr{E}) \to F$. The problem is that $\mathbb{P}(\mathscr{E}) \to X$ may no longer be of finite type.

Applying the construction of expanded degenerations (I.4.23) to the data $(U, \mathbb{P}(\mathscr{E}), Z =$ reduction of $\mathbb{P}(\mathscr{E}))$, we obtain a morphism $U \subseteq U^{\text{él}} \to F$. By proposition I.4.27, $U^{\text{él}} \to X$ is overconvergent. □

In other words, for any X,

$$\mathbf{Rig}_X^{\mathrm{sur}} \subseteq \mathrm{Sh}\mathbf{Rig}_X^{\mathrm{sur}}$$

is a spatial geometric context as in definition I.1.1 whose category of locally representable objects is the *large overconvergent site* $\mathbf{Rig}_X^{\mathrm{sur}}$ of X.

Overconvergence being stable for composition and base change, a morphism $f : X \to Y$ naturally induces an essential spatial geometric morphism

$$f : \mathrm{Sh}\mathbf{Rig}^{\mathrm{sur}}_X \to \mathrm{Sh}\mathbf{Rig}^{\mathrm{sur}}_Y \qquad f_! : \mathbf{Rig}^{\mathrm{sur}}_X \leftrightarrows \mathbf{Rig}^{\mathrm{sur}}_Y : f^*$$

which, by locality on the base, makes $\mathrm{Sh}\mathbf{Rig}^{\mathrm{sur}}_-$ and $\mathbf{Rig}^{\mathrm{sur}}_-$ into *stacks* on **Rig**; the latter is locally a site for the former.

Small site The terminal object X of $\mathbf{Rig}^{\mathrm{sur}}_X$ has its own small topos $\mathrm{Sh}(X^{\mathrm{sur}}) = \mathrm{Sh}(\mathscr{U}^{\mathrm{sur}}_{/X})$ which, although incoherent, is a subtopos of $\mathrm{Sh}(X)$ and therefore has enough points. In particular, it is *spatial*, with determining sober topological space (or, if you prefer, locale) X^{sur}. The geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(X^{\mathrm{sur}})$ induces a surjective continuous mapping

$$b : X \to X^{\mathrm{sur}}$$

which the literature has called the *separation map* ([FK13, I.2.4.(c)]). A morphism $f : X \to Y$ yields, by restriction from the large overconvergent topos, a natural geometric morphism $X^{\mathrm{sur}} \to Y^{\mathrm{sur}}$ such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "b"'] & Y \arrow[d, "b"] \\ X^{\mathrm{sur}} \arrow[r, "f"] & Y^{\mathrm{sur}} \end{tikzcd}$$

commutes.

By definition, the square

$$\begin{tikzcd} \mathscr{U}^{\mathrm{sur}}_{/X} \arrow[r] \arrow[d, "b^{-1}"'] & \mathbf{Rig}^{\mathrm{sur}}_X \arrow[d] \\ \mathscr{U}_{/X} \arrow[r] & \mathbf{Rig}_X \end{tikzcd}$$

is Cartesian. We would also like to know when the extended square

$$\begin{tikzcd} \mathrm{Sh}X^{\mathrm{sur}} \arrow[r] \arrow[d, "b^*"'] & \mathrm{Sh}\mathbf{Rig}^{\mathrm{sur}}_X \arrow[d] \\ \mathrm{Sh}X \arrow[r] & \mathrm{Sh}\mathbf{Rig}_X \end{tikzcd}$$

is also Cartesian, so that every sheaf on the small site of X that is overconvergent as an object of $\mathrm{Sh}\mathbf{Rig}_X$ is actually the pullback under b of a sheaf on X^{sur}.

11.16 Lemma. *Suppose that X is purely analytic. Then* $\mathrm{Sh}(X^{\mathrm{sur}})$ *is the full subcategory of* $\mathrm{Sh}(X)$ *whose objects are overconvergent.*

Proof. Follows as in the proof of lemma 11.15, with $U \subseteq X$ a quasi-compact open subset. Since X is purely analytic, $U^{\mathrm{él}} \to X$ is an open immersion. □

11.17 Example. This statement is false for formal schemes. For instance, any non-simply-connected Noetherian formal scheme has finite covering spaces, but no non-trivial overconvergent open subsets over which to find a section.

11.7 Local compactness

Overconvergent open sets are almost never quasi-compact. For practical reasons, it is often easier to work with quasi-compact objects; hence the following definition:

11.18 Definitions (Overconvergent coverings)**.** A covering in $\mathbf{Rig}_X$ is *overconvergent* if it can be refined by a covering in $\mathbf{Rig}_X^{\mathrm{sur}}$.

An analytic space X is said to be *overconvergent-locally compact* if every qcqs open subset of X admits a qcqs overconvergent neighbourhood. It is *(overconvergent-)locally convex* if the neighbourhood can always be taken *affine*. We usually abuse notation by omitting the prefix 'overconvergent'.

Locally compact analytic spaces over $\mathbb{Z}$ were called variously *locally quasi-compact* and *strongly locally compact* in [FK13, II.4.4.1]; locally convex spaces are what Berkovich calls *good* [Ber93].

Any qcqs space is locally compact, and any affine analytic space is locally convex. Local convexity is usually only a reasonable condition in the non-affine case when the overconvergent topology is reasonable, that is, when the space is purely analytic.

11.19 Lemma. *A paracompact analytic space is locally compact.*

Proof. By choosing a model we may reduce to the case of formal schemes. Let X be a paracompact formal scheme, $U_0 \subseteq X$ be an affine open immersion. Since X is paracompact, $\mathrm{cl}(U_0/X)$ is quasi-compact. □

11.20 Lemma. *Let S be qcqs, X overconvergent over S. Then X is locally compact.*

Proof. Let $U \hookrightarrow X$ be an affine open subset. Then $U \to S$ is of finite type, and so may be immersed into some projective bundle $\mathbb{P}(\mathscr{E})/S$. By overconvergence, there is a formally projective morphism $\tilde{\mathbb{P}}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ and an extension

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \tilde{\mathbb{P}}(\mathscr{E}) \arrow[d] \arrow[dl, dashed] \\
X \arrow[r] & S
\end{tikzcd}$$

and by part *v)* of proposition I.4.6, $\tilde{\mathbb{P}}(\mathscr{E}) \to X$ is $f\mathbb{P}$-proper, and hence an overconvergent neighbourhood of U. □

11.21 Proposition. *Let $X \in$ **Rig** be purely analytic and locally compact. Then:*

i) X admits an overconvergent covering by qcqs open sets;

ii) X admits a covering by quasi-separated overconvergent open sets.

In particular, a morphism $X \to S$ is overconvergent if and only if it is overconvergent on every quasi-separated overconvergent open subset.

Proof. First note that *i)* immediately implies *ii)* by the definition of overconvergent cover and the fact that any open subset of a quasi-separated space is quasi-separated. The proof of *i)* follows from the construction of expanded degenerations (I.4.23). □

11.22 Aside. Via the theory of collages exhibited in §12.3, it is easy enough to find examples of analytic spaces that are locally of finite type over $\mathbb{F}_1((t))$ but not locally compact; indeed, any non-locally-compact subset of $\mathbb{R}^n$ that is exhausted by rational polyhedra will do the trick.

In fact, this type of approach seems to permit the construction of toric analytic spaces with any egregious topological property imaginable.

11.8 Hausdorff quotient

Under a certain technical assumption on an analytic space X, proposition 11.14 has point-set-topological consequences.

(CL) The formally embedded closure of a quasi-compact open immersion into X is closed.

It is equivalent that this be true for a coinitial family of models of X, since X is weakly topologised with respect to such a family. Condition (CL) is manifestly satisfied by all analytic spaces over $\mathbb{Z}$ and *quasi-integral* analytic spaces over $\mathbb{F}_1$.

11.23 Aside. Let $A \to A[f^{-1}]$ be a localisation of discrete $\mathbb{F}_1$-algebras. The precise condition that A, f must satisfy for the embedded closure of $\operatorname{Spec} A[f^{-1}]$ to be closed in $\operatorname{Spec} A$ is that the action of f on $A/\operatorname{Ann}(f)$ be *injective*. Therefore, for example, any $\mathbb{F}_1$-algebra with non-trivial idempotents will fail our condition.

Under condition (CL) and by corollary 11.14, an open subset U of X is overconvergent if and only if, for every quasi-compact open immersion factoring through U, the point set topological closure is also in U.

If X is quasi-separated, then by [FK13, I.2.3.4] it is enough that the closure of every *point* of U is in U; thus, our definition is equivalent to *op. cit.* I.2.4.10. In the case of purely analytic spaces (def. 10.4), the arguments of *op. cit.* §II.4.1 apply to show that we are in the context of what the authors call 'valuative' spaces. We may therefore apply the results of *op. cit.* §I.2.4.(d).

11.24 Aside. If X is a locally Noetherian topological space, then X^{sur} is just a single point. This includes most interesting formal schemes. A similar statement applies to any connected rigid space containing a formal scheme. As such, the overconvergent topology is only likely to be interesting for purely analytic rigid spaces (def. 10.4).

11.25 Theorem (Properties of the overconvergent topology)**.** *Let X be a rigid analytic space satisfying the condition* (CL) *- for instance, any analytic space over $\mathbb{Z}$ or any quasi-integral analytic space over $\mathbb{F}_1$. Then:*

i) X^{sur} is compactly generated and T1;

ii) if X is overconvergent-locally compact (def. 11.18), then X^{sur} is locally compact.

If X is moreover purely analytic and quasi-separated, then X^{sur} is a universal Hausdorff quotient of X.

Proof. Compact generation. A rigid analytic space is topologically a colimit of quasi-compact open subsets, and overconvergence is detected on each subset. We will see shortly that in fact, affine subsets become compact Hausdorff.

T1. A space is T1 if and only if points are closed, so we have to prove that points of X related by specialisation are topologically indistinguishable in X^{sur}. Let $x \in \overline{\{y\}}$. Then every open neighbourhood of x contains y. In particular, there is a qcqs open set X_0 containing both points. Now let $U \subseteq X$ be an overconvergent open neighbourhood of Y. By pulling back to X_0, we may assume the ambient space is quasi-separated. There is a qcqs neighbourhood of y contained in U. By corollary 11.14, its closure, and in particular x, is contained in U.

Locally compact. Clear from the definition.

Hausdorff. This is [FK13, I.2.5.8], which we may apply because our purely analytic space is 'valuative'. For the universal property, let $q : X \to K$ be any continuous map into a Hausdorff space. It will suffice to show that q^{-1} takes open sets of K to overconvergent sets of X. We may assume X is affine and hence that K is quasi-compact. Let $V \subseteq K$ be open, $U \subseteq q^{-1}V$ quasi-compact. Then $qU \subseteq K$ is compact and hence closed. Therefore the embedded closure of U is contained in $q^{-1}V$. Thus $q^{-1}V$ is overconvergent. □

12 From rigid spaces to affine manifolds

In this section we will be interested in normal rigid analytic spaces locally of finite type over a valuation $\mathbb{F}_1$-field $K = \mathbb{F}_1(\!(t^{-H})\!)$, with ring of integers $\mathcal{O}_K = \mathbb{F}_1[\![t^{-H}]\!]$, for $H \subseteq \mathbb{R}$ an additive subgroup of the reals.

12.1 Convergence polyhedron

In the opening sentences §9, we made some intuitive remarks about strongly convex polyhedra defined by inequalities over H inside an H-affine space N. These assemble to form a category $\mathbf{Poly}_H^N$ of embedded polyhedra, with morphisms affine maps of the ambient affine spaces that preserve the polyhedra. Our convention will be that $\emptyset$ is *not* a polyhedron.

An object Δ of $\mathbf{Poly}_H^N$ is determined by the pair

$$\mathrm{Aff}_\Delta^+(N,H) \subseteq \mathrm{Aff}_\Delta(N,H),$$

which itself determines a Banach K-algebra

$$\mathbb{F}_1(\!(t^{-H})\!) \to \mathcal{O}\{\Delta\} = \left(\mathbb{F}_1\left\{z^{\mathrm{Aff}_\Delta(N,H)}\right\}; \mathbb{F}_1\left\{z^{\mathrm{Aff}_\Delta^+(N,H)}\right\}\right)$$

defined, as usual, by writing $\mathrm{Aff}_\Delta^+(N,H) \subseteq \mathrm{Aff}_\Delta(N,H)$ multiplicatively, adjoining 0, and equipping it with the t-adic topology. It is automatically normal (since Aff_Δ is saturated) and of finite type over K.

This construction is natural in Δ and hence determines a fully faithful functor

$$X : \mathbf{Poly}_H^N \to \mathbf{Rig}^{\mathrm{aff/tf/n/nb}}_{\mathbb{F}_1(\!(t^{-H})\!)}, \quad \Delta \mapsto X_\Delta$$

into the category of affine and normal rigid analytic spaces of finite type over $\mathbb{F}_1(\!(t^{-H})\!)$ with non-boundary morphisms (that is, whose dual $\mathbb{F}_1$-algebra homomorphisms have no kernel).

Conversely, a finitely presented, quasi-integral Banach K-algebra A gives rise to a finite-dimensional affine space

$$N_{A/H}(-) = \mathrm{Hom}_H(K_A^\times, -),$$

which we consider as a functor on rank one extensions of H. Adapting our previous practice, write $\log f$ for the affine function on N determined by $f \in K_A^\times$. Then

$$\Delta_{A/H} = (\log f \leq 0 | f \in A^+ \setminus 0) \subseteq N_{A/H}$$

is a strongly convex polyhedron, the *convergence polyhedron* of A (cf. [EKL07, 3.1.4]). It depends only on the relative normalisation of $(A;A^+)$, and hence descends to a functor

$$\Delta : \mathbf{Rig}^{\mathrm{aff/tf/qi/nb}}_{\mathbb{F}_1((t^{-H}))} \to \mathbf{Poly}^N_H, \quad \operatorname{Spec} A \mapsto \Delta_{A/H}$$

right adjoint to X.

12.1 Lemma. *The adjunction* $\mathbf{Poly}^N_H \leftrightarrows \mathbf{Rig}^{\mathrm{aff/tf/qi/nb}}_{\mathbb{F}_1((t^{-H}))}$ *restricts to an equivalence on the full subcategory of* $\mathbf{Rig}^{\mathrm{aff/tf/qi/nb}}_{\mathbb{F}_1((t^{-H}))}$ *whose objects are normal with no torsion in* $K^\times$.

12.2 Aside. We can fix the issue with torsion in $K^\times$ by considering our polyhedra equipped with an Abelian group $K^\times$ that pairs with N; to avoid unnecessary complication, I will instead make the hypothesis that $K^\times$ is torsion-free a standing hypothesis from now on.

12.3 Example (Field extensions). If H' is a degree n extension of H, then $\Delta_{H'/H}$ can be realised as a 0-dimensional polyhedron inside the n-dimensional H-affine space H' whose vertex is a generator of H' over H.

Open sets In a completed localisation of Banach K-algebras (cf. §10.4), we are allowed to invert some elements $s \in A \setminus 0$ and then enlarge A^+ by throwing in some elements of the form $t/s \in A[s^{-1}]$. This corresponds to passing to the *sub-polyhedron* of $\Delta_{A/H}$ defined by the inequalities $\log t \leq \log s$.

Thus affine open subsets of $\operatorname{Spec} A$ correspond to sub-polyhedra of $\Delta_{A/H}$.

Points If H' is an extension of H, then a non-boundary $\mathbb{F}_1((t^{-H'}))$-point of $\operatorname{Spec} A$ over K is a commuting diagram

$$\begin{tikzcd}
H^\circ \arrow[r] \arrow[d] & \mathrm{Aff}^+_{\Delta_{A/H}}(N_{A/H}, H) \arrow[r] \arrow[d] & (H')^\circ \arrow[d] \\
H \arrow[r] & \mathrm{Aff}_{\Delta_{A/H}}(N_{A/H}, H) \arrow[r] & H'
\end{tikzcd}$$

where the top and bottom horizontal compositions are the structural map $H \subseteq H'$; in other words, it is an element of $\Delta_{A/H}(H')$.

These points can actually be realised as morphisms of polyhedra; see example 12.3.

Convergence region The passage from $\Delta_{A/H}$ to $N_{A/H}$ forgets the topology (and ring of integers) of A: there is a Cartesian square

$$\begin{tikzcd}
\Delta_{A/H}(-) \arrow[r] \arrow[d] & \mathrm{Hom}_K\left(A, \mathbb{F}_1((t^{-(-)}))\right) \arrow[d] \\
N_{A/H}(-) \arrow[r] & \mathrm{Hom}_{K^?}\left(A^?, \mathbb{F}_1((t^{-(-)}))\right)
\end{tikzcd}$$

As all functions extend meromorphically over arbitrary expansions of $\Delta_{A/H}$, the term 'convergence' here is purely in analogy with the case of analysis over topological fields.

12.4 Aside. Over an ordinary non-Archimedean field, any boundaryless rigid analytic space will be either an algebraic field extension or of infinite type. The existence of geometrically interesting boundaryless rigid spaces of finite type over a field is therefore a peculiarity of the $\mathbb{F}_1$-world.

In the boundaryless case, the points calculation simplifies to

$$\Delta_{A/H}(H') \cong \mathrm{Hom}_K\left(A, \mathbb{F}_1(\!(t^{-H'})\!)\right)$$

that is, $\Delta_{A/H}$ is, as a functor, simply the restriction of $\mathrm{Spec}A$ to the category of rank one extensions of H. Of course, it is possible to give a combinatorial description of the boundary as well (§9.4), but this is hardly more straightforward than the definition of $\mathrm{Spec}A$ as a functor on Banach K-algebras.

12.2 Formal models

Let $X = \mathrm{Spec}A \in \mathbf{Rig}_K^{\mathrm{aff/tf/qi/nb}}$ and let X^+ be a relatively normal formal model of X. By definition,

$$\mathrm{Hom}_K(\Delta_{H'} \setminus\!\!\setminus \{0\}, X) = \mathrm{Hom}_{\mathscr{O}_K}(\Delta_{H'}, X^+)$$

with $\Delta_H = \mathrm{Spec}\mathbb{F}_1[\![t^{-H'}]\!]$ the formal disc with exponent group $H' \supseteq H$. If $\mathscr{O}_{X^+}$ is t-torsion-free, it is in particular quasi-integral and so we can define the punctured cone complex Σ_{X^+} and its developing map $\Sigma_{X^+} \to N_{X^+} = \mathrm{Hom}(K_X^\times, -)$. The above identification yields get affine inclusions

$$\begin{tikzcd}
\Delta_{X/H} \arrow[r, hook] \arrow[d] & \Sigma_{X^+} \arrow[d] \\
N_{X/H} \arrow[r, hook] & N_{X^+}
\end{tikzcd}$$

as the fibre over the identity of the restriction map $\mathrm{Hom}(K_X^\times, H) \to \mathrm{End}(H)$. (If $H \not\subseteq \mathbb{Q}$, the objects in the right column may be replaced with their relative variants discussed at the end of §I.7.2.)

If we pick $X^+ = \mathrm{Spec}A^+$ the canonical affine model of X, Σ_{X^+} will simply be the cone over $\Delta_{X/H} \subseteq \mathrm{Hom}(K_X^\times, -)$, punctured along the kernel of $\mathrm{Hom}(K_X^\times, -) \to \mathrm{Hom}(H, -)$. In general it will be a finite punctured fan whose support is this cone.

Intersecting Σ_{X^+} with $\Delta_{X/H}$ decomposes it into H-rational convex bodies. By compactness, any such decomposition must in fact be into finitely many H-rational polyhedra.

12.5 Proposition. *The category of relatively normal models of X is equivalent to the poset of polyhedral decompositions of $\Delta_{X/H}$, ordered by refinement.*

Although we already observed this through algebra, this gives a geometric proof that:

12.6 Corollary. *The convergence polyhedron functor Δ sends non-empty open immersions to inclusions of polyhedra.*

12.3 Collages

Glueing The functor $\mathbf{Rig}_K^{\text{aff/tf/qi/nb}} \to \mathbf{Poly}_H^N$ is left exact and creates limits, and therefore flat. Unlike previously, however, we now have non-trivial coverings in $\mathbf{Rig}_K^{\text{aff/tf/qi/nb}}$, so we will need to introduce some compatibility in order to globalise our constructions.

Fortunately, it is possible to understand these coverings purely in terms of the points valued in the maximal totally ramified extension $K^{\text{ram}} = \mathbb{F}_1((t^{-\mathbb{Q}H}))$ of K (here $\mathbb{Q}H \subseteq \mathbb{R}$ denotes the divisible hull of H).

12.7 Lemma. *A finite family of affine open subsets $U_i \subseteq X$ is a covering if and only if $X(K^{\text{ram}}) = \bigcup_i U_i(K^{\text{ram}})$.*

Proof. We may find a model of X^+ of X on which each $U_i \hookrightarrow X$ is realised as an open immersion. The covering condition then becomes that Σ_{X^+} is a union of cones in the subfans $\Sigma_{U_i^+}$; this is detected by the rational points $\Sigma_-(\mathbb{Q}H)$. (This would be false if we allowed an infinite family of U_i.) □

Defining coverings on $\mathbf{Poly}_H^N$ to be those finite families of sub-polyhedra that induce a surjection on $\mathbb{Q}H$-rational points, we obtain a sheaf topos $\mathrm{Sh}\mathbf{Poly}_H^N$ and full subcategory $\mathrm{C}\mathbf{Poly}_H^N$ of locally representable objects.

12.8 Definition. An object of $\mathrm{C}\mathbf{Poly}_H^N$ is called a *collage in embedded H-rational polyhedra*, or simply *collage* if the constituent objects are understood.

12.9 Proposition. *The convergence polyhedron functor Δ extends to the pullback along a geometric morphism*

$$\mathrm{Sh}\mathbf{Poly}_H^N \to \mathrm{Sh}\mathbf{Rig}^{\text{ltf/qi/nb}}_{\mathbb{F}_1((t^{-H}))}$$

which preserves open immersions and induces bijections on open subset lattices.

It has a fully faithful left adjoint with image the full subcategory generated under colimits by the normal analytic spaces.

12.10 Corollary. *The convergence region functor and its left adjoint restrict to an equivalence*

$$\Delta : \mathbf{Rig}^{\text{ltf/n/nb}}_{\mathbb{F}_1((t^{-H}))} \xrightarrow{\sim} \mathrm{C}\mathbf{Poly}_H^N$$

between the category of normal analytic spaces and the category of collages.

A family of open subsets $U_i \subseteq X$ is a covering if and only if on every polyhedron Δ of $\Delta_{X/H}$ there is a finite refinement such that $\Delta(\mathbb{Q}H) = \bigcup_i \Delta \cap \Delta_{U_i/H}(\mathbb{Q}H)$.

12.11 Example (Affine space). An H-affine space N can be considered as a collage

$$N(\Delta) = \mathrm{Hom}(\Delta(H_\infty), N(H))$$

(which is empty unless $\Delta(H_\infty) = \Delta(H)$ is bounded). Given a strongly convex cone σ in $\Lambda_{N/H}$, one can also define a partial compactification by allowing morphisms from infinite polyhedra whose recession cone is contained in σ.

Developing map A collage in bounded polyhedra comes equipped with a locally (i.e. on each polyhedron) defined *developing morphism*

$$\delta : \Delta \to N$$

which extends globally on any universal cover. A collage in possibly unbouded polyhedra still has local developing morphisms into varying partial compactifications of N.

The topological realisation $\Delta(\mathbb{R})$ of a locally compact collage Δ comes equipped with a local system of real affine spaces $N(\mathbb{R})$ with H-structures and a canonical section

$$\Delta(\mathbb{R}) \to N(\mathbb{R})$$

rendering $N(\mathbb{R})$ a real vector bundle. In nice cases (cf. prop. 12.18), the developing map $\delta : \Delta \to N_p$ at p will be defined on a Euclidean open neighbourhood of p, but this fails in general.

If X has a model X^+, then the developing map is obtained as the fibre over $1_{\mathrm{End}(H)}$ of the developing map associated to the punctured cone complexes $\Sigma_{X^+}(\mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(K_X^\times, \mathbb{R})$. More generally, X admits models quasi-compact-locally, and so $\Delta_X(\mathbb{R}) \to N(\mathbb{R})$ is a topological filtered colimit of sections of punctured cone complexes.

12.4 Overconvergent topology of polyhedra

The complex $\Delta_{X/H}(\mathbb{R}_\infty)$ gives a neat description of the *overconvergent topology* of X: the natural map $c : \Delta_{X/H}(\mathbb{R}_\infty) \to X$ induces a pullback on open sets of X

$$c^{-1} : \mathscr{U}_{/X} \to \mathscr{P}(\Delta_{X/H}(\mathbb{R}_\infty)) := \{\text{subsets of } \Delta_{X/H}(\mathbb{R}_\infty)\}, \quad U \mapsto U(\mathbb{R}_\infty)$$

that matches the overconvergent sets one-to-one with the Euclidean open subsets.

12.12 Theorem (Points of the overconvergent topos)**.** *Let X be a quasi-integral rigid space, locally of finite type over $\mathbb{F}_1((t^{-H}))$. The composite of the natural map $c : \Delta_{X/H}(\mathbb{R}_\infty) \to X$ with the separation map b induces a homeomorphism between $\Delta_{X/H}(\mathbb{R}_\infty)$ and X^{sur}.*

The proof of this statement occupies the rest of this section.

The inverse to our pullback c^{-1} will come from its right adjoint

$$c_* : \mathscr{P}(\Delta_X(\mathbb{R}_\infty)) \to \mathscr{U}_{/X} \quad c_*S = \bigcup_{\Delta'(\mathbb{R}) \subseteq S} \Delta'$$

which takes a subset $S \subseteq \Delta_X(\mathbb{R}_\infty)$ to the colimit of all H-rational polyhedra whose topological realisation it contains.

Since every Euclidean open set is a union of rational polyhedra:

12.13 Lemma. *Let $S \subset \Delta_X(\mathbb{R}_\infty)$ be Euclidean open; then $c^{-1}c_*S = S$.*

It remains to show that the essential image of the restriction of c^{-1} to $\mathscr{U}^{\mathrm{sur}}_{/X}$ consists of Euclidean open sets.

Applying proposition I.7.14 to any model of X immediately yields a characterisation:

12.14 Proposition. *Let X be an affine, quasi-integral rigid analytic space of finite type over K, $U \subseteq V \subseteq X$ quasi-compact open subsets. Then V is an overconvergent neighbourhood of U if and only if $\Delta_{V/H}(\mathbb{R}_\infty)$ is a neighbourhood of $\Delta_{U/H}(\mathbb{R}_\infty)$ in $\Delta_{X/H}(\mathbb{R}_\infty)$.*

Combining this and corollary 11.14:

12.15 Corollary. *An open subset $U \subseteq X$ is overconvergent if and only if $\Delta_{U/H}(\mathbb{R}_\infty)$ is open in $\Delta_{X/H}(\mathbb{R}_\infty)$.*

This completes the proof of theorem 12.12 for affine X. The global statement is a straightforward generalisation: there is an adjunction

$$c^{-1} : \mathcal{U}_{/X} \leftrightarrows \mathcal{P}(\Delta_X(\mathbb{R}_\infty)) : c_*$$

with c^{-1} left exact, and the overconvergent sets (resp. open sets) on the left (resp. right) are exactly those that remain so after pullback to affine set (resp. polyhedron). Therefore $c^{-1} \dashv c_*$ restrict to an equivalence $\mathcal{U}^{\mathrm{sur}}_{/X} \cong \mathcal{U}_{\Delta_X(\mathbb{R})}$.

12.5 Overconvergence criteria

The arguments of this section are largely modelled on §I.7.5 - strangely, passing to analytic geometry actually introduces simplifications.

12.16 Lemma. *Let Δ be an overconvergent-locally compact collage. Then $\Delta(\mathbb{R})$ is a locally contractible topological space.*

Proof. The statement is straightforward in the case Δ is qcqs. More generally, if Δ is locally compact, then every polyhedron Δ_U has a qcqs overconvergent neighbourhood Δ_V. By proposition 12.14, $\Delta_V(\mathbb{R})$ restricts to a Euclidean neighbourhood of $\Delta_U(\mathbb{R})$ on each polyhedron of Δ. Since $\Delta(\mathbb{R})$ is strongly topologised by polyhedra, the result follows. □

12.17 Aside. The converse is false, because in certain non-locally-compact geometries, $\Delta(\mathbb{R})$ may fail to detect topological features. For instance, let N be a 2-dimensional $\mathbb{Z}$-affine space, considered as a non-compact collage as in example 12.11, and let $P_i \subset N$ be a filtered family of rational polyhedra such that

- each P_i has non-empty interior;
- $\bigcap_i P_i = \{x\}$ with x a vertex of each P_i.

Then

$$U := \bigcup_i (N \setminus \mathrm{int}(P_i))$$

is an open subset of N with the same set of $\mathbb{R}$-points ($\simeq \mathbb{R}^2$). In particular, it is separated. However, the bijection $U(\mathbb{R}) \to N(\mathbb{R})$ is not coming from a covering that is locally finite at x, and so by lemma 12.7 $U \hookrightarrow N$ is not an isomorphism. In fact, U is not locally simply-connected at x.

In particular, the developing map of a locally compact collage extends to a qcqs overconvergent neighbourhood of any polyhedron. This allows us to apply the arguments of propositions I.7.19 and I.7.23 to prove analogous statements for analytic spaces:

12.18 Proposition. *Let $\Delta \in \mathrm{C}\mathbf{Poly}_H^N$ be a locally compact collage. Then the developing map $\delta : \Delta \to N$ is overconvergent-locally defined. Moreover:*

i) *Δ is locally separated if and only if δ is an overconvergent-local immersion;*

ii) *Δ is overconvergent if and only if δ is an overconvergent-local homeomorphism.*

Proof. Since the arguments are essentially the same as those of *loc. cit.*, I include only one part, by way of illustration.

Suppose that Δ is overconvergent, and let $\Delta_U \subseteq \Delta$ be a polyhedron. There is a qcqs overconvergent neighbourhood Δ'_U of Δ_U such that $\delta : \Delta'_U \to N_U$ is defined. Let $\Delta_V \subseteq N_U$ be a polyhedron whose real points contain a neighbourhood of $\Delta_U(\mathbb{R}_\infty)$. By overconvergence, after possibly shrinking Δ_V, there is a unique section

$$\Delta_V \to \Delta'_U$$

to δ. Thus δ is a local homeomorphism. □

For simplicity, I have stated the absolute version, though to derive the relative version as in I.7.23 would be straightforward. Note that unlike the case of punctured cone complexes, the collages appearing in this theorem are not necessarily quasi-separated.

12.19 Corollary. *An H-affine space N, considered as a collage (e.g. 12.11), is overconvergent. The natural map $N(\mathbb{R}) \to N^{\mathrm{sur}}$ is a homeomorphism.*

It follows from theorem 12.12 that in fact:

12.20 Corollary. *Let $\Delta \in \mathrm{C}\mathbf{Poly}_H^N$ be a locally compact collage in bounded polyhedra. Then the developing map $\delta : \Delta(\mathbb{R}) \to N(\mathbb{R})$ is locally defined. Moreover:*

i) *Δ is locally separated if and only if δ is an local immersion;*

ii) *Δ is overconvergent if and only if δ is an local homeomorphism.*

12.21 Corollary. *Let X be a quasi-integral analytic space locally compact and locally of finite type over $\mathbb{F}_1((t^{-H}))$. The developing map*

$$\delta : \Delta_{X/H} \to N_{X/H}$$

is locally defined. Moreover:

i) *X is locally separated if and only if δ is a local immersion;*

ii) *X is overconvergent if and only if δ is a local homeomorphism.*

12.22 Corollary. *Let X be overconvergent. Then $\Delta_X(\mathbb{R})$ carries a unique structure of an affine manifold with developing map δ.*

12.6 H-affine manifolds as rigid analytic spaces over $\mathbb{F}_1((t^{-H}))$

Here we globalise the constructions of §12.4. For simplicity, we will treat only the *boundaryless* rigid analytic spaces. One can classify more general normal analytic spaces in terms of affine manifolds with corners; a sketch-definition of such objects can be found in §9.6.

12.23 Definition. A rigid analytic space is said to be *boundaryless* if its only closed subspaces are unions of connected components. Equivalently, it is locally modelled by the spectra of locally convex $\mathbb{F}_1$-fields.

Note that the condition of being boundaryless in particular implies normality. It does not say anything about the topological boundary of $\Delta(\mathbb{R})$.

Let $\mathbf{Aff}_H$ be the category of paracompact H-affine manifolds and H-affine maps. It is finitely complete, and generated under colimits by open subsets of H-affine space. The Euclidean topology endows $\mathbf{Aff}_H$ with the structure of a spatial, but not coherent, site. If we allow affine manifolds to be non-Hausdorff, then Yoneda matches $\mathbf{Aff}_H$ with the category of all paracompact locally representable objects of $\mathrm{Sh}\mathbf{Aff}_H$.

Every object of $\mathbf{Aff}_H$ is exhausted by bounded polyhedra, and so the functor

$$c^{-1} : \mathbf{Aff}_H \to \mathrm{Sh}\mathbf{Poly}_H^N, \quad U \mapsto \left[\Delta \mapsto \begin{cases} \mathrm{Hom}_H(\Delta, U) & \Delta \text{ bounded} \\ \emptyset & \text{otherwise} \end{cases} \right]$$

is fully faithful.

12.24 Lemma. *c^{-1} preserves finite limits, open immersions, and coverings.*

Proof. Lex. An H-affine map $\Delta \to U$ is determined by its underlying map of sets $\Delta(H) \to U(H)$, and affineness is stable for fibre products. Therefore c^{-1} is left exact.

Open immersions. Clear from the definition.

Coverings. Let $U = \bigcup_i U_i$ be an open covering. Since U is paracompact, the cover may be assumed locally finite. It therefore restricts to a locally finite family on each polyhedron that is covering for $\mathbb{R}$-points, and hence a covering by lemma 12.7. □

It therefore extends to a geometric morphism

$$c : \mathrm{Sh}\mathbf{Poly}_H \to \mathrm{Sh}\mathbf{Aff}_H, \qquad c^* : \mathbf{Aff}_H \to \mathrm{C}\mathbf{Poly}_H$$

whose pullback preserves locally representable objects. It also respects developing maps. By the criterion of proposition 12.18, it follows that the objects in the essential image of c^* are actually *overconvergent*. By construction, they also do not admit morphisms from any collage with boundary, and so are themselves boundaryless.

12.25 Theorem. *The geometric morphism*

$$c : \mathrm{Sh}\mathbf{Poly}_H^{\partial/\mathrm{sur}} \to \mathrm{Sh}\mathbf{Aff}_H$$

is a spatial equivalence of categories.

Proof. We have already seen that c^* is fully faithful. It remains to show that the image of $\mathbf{Aff}_H$ is a site for the overconvergent topos. This is a consequence of corollary 12.18 and the fact that $\mathbf{Aff}_H$ generates the overconvergent topology of any H-affine space N. □

By composing all our functors, we obtain the fully faithful functor

$$\mathbf{Aff}_H \to \mathbf{Rig}_K^{\partial/\mathrm{sur}}$$

that is the title of this document. The objects in the essential image of this functor are paracompact, quasi-separated, overconvergent, and boundaryless (which for obvious reasons, I have chosen not to record in the superscript).

It realises an affine manifold B as a rigid analytic space B^{rig}, together with a (discontinuous) map $c : B \to B^{\mathrm{rig}}$ that descends to a homeomorphism on the overconvergent site.

12.26 Corollary. *The restriction of c^* is a topological equivalence*

$$\mathbf{Aff}_H \cong \mathbf{Rig}^{\partial/\mathrm{sur}}_{\mathbb{F}_1((t^{-H}))}$$

between the category of (not necessarily Hausdorff or paracompact) H-affine manifolds and the category of boundaryless, overconvergent, rigid analytic spaces.

Moreover, the following are equivalent:

i) *X has affine diagonal;*

ii) *X is quasi-separated;*

iii) *$\Delta_X(\mathbb{R})$ is Hausdorff.*

Proof. Suppose that $\Delta(\mathbb{R})$ is Hausdorff, and let $\Delta_1, \Delta_2 \subseteq \Delta$ be two polyhedra. By the Hausdorff property, $\Delta_i(\mathbb{R})$ is closed in $\Delta(\mathbb{R})$.

There are overconvergent neighbourhoods U_i of Δ_i such that $\delta : U_i \hookrightarrow N(\mathbb{R})$ is an open immersion. The intersection $\Delta_2 \cap U_1$ is closed in U_1 and contains $\Delta_1 \cap \Delta_2$. It follows that the latter can be realised as an intersection of two polyhedra inside an affine space, and is therefore a polyhedron. This proves that Δ has affine diagonal. □

12.27 Example (Groups). The inclusion $\mathbf{Rig}_K^{\mathrm{ltf/n/nb}} \hookrightarrow \mathbf{Rig}_K$ being left exact, it follows that an affine manifold with the structure of a *group* induces a group structure on its rigid analytic space. Many motivating examples of toric rigid analytic spaces arise in this way.

For example, the rigid space associated to the affine manifold $H \subseteq \mathbb{R}$ is a group object $\mathbb{G}_m/K$ of $\mathbf{Rig}_K$, the *multiplicative group* over K. More generally, to the affine manifold $\mathrm{Hom}(\Lambda, H)$, where Λ is any lattice, one puts $\mathrm{Hom}(\Lambda, \mathbb{G}_m)$ for the *diagonalisable group with character group* Λ. Note that in contrast to the algebraic setting, these diagonalisable groups are not quasi-compact.

12.7 On the Mumford degeneration

Continuing on from the previous example, let us investigate proper, commutative group objects of $\mathbf{Rig}_{\mathbb{F}_1((t^{-H}))}$. Under the correspondence of corollary 12.26, these are nothing more than compact H-affine group manifolds.

Let B be an H-affine group manifold. The model affine space N_B at the origin is naturally a vector space. We will assume that B is connected and complete, that is, that the developing map $\widetilde{B} \to N_B$ is an isomorphism. (The latter is probably automatic for compact affine manifolds with integer slopes.)

Note that here, as above (remark 12.2), we are also implicitly restricting attention to the case where $K_B^\times$ is torsion-free. In geometric terms, this means that a base change to a field of characteristic zero is connected. Let us call this property *geometrically connected*.

In this case, the universal covering gives a uniformsation

$$B \cong N_B/Y$$

with $Y = \pi_1(B,0)$ a cocompact, discrete subgroup of $N_B(H)$; that is, a lattice. In particular, B is an affine torus with circumferences in H.

This uniformisation translates onto the rigid analytic side; the covering N_B corresponds, as in example 12.27, to a torus $T = N_B \otimes \mathbb{G}_m$ with character group $X^*(T) = \Lambda_{B/H}$, and Y to a subgroup of the $\mathbb{F}_1((t^{-H}))$-points of T. We get the following commutative square of equivalent categories:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{proper, geometrically connected,}\\ \text{commutative groups in } \mathbf{Rig}_{\mathbb{F}_1((t^{-H}))}\end{array}\right\} \arrow[r, dash, "\text{cor. 12.26}"] \arrow[d, dash] & \{H\text{-affine tori }\} \arrow[d, dash] \\
\left\{\begin{array}{c}\text{pairs } (T,Y) \text{ with } T \text{ an algebraic torus over}\\ \mathbb{F}_1((t^{-H})) \text{ and } Y \subseteq T(\mathbb{F}_1((t^{-H}))) \text{ a lattice with}\\ \mathrm{rk}(Y) = \mathrm{rk}(T) \text{ and } \mathcal{O}(T) \to \mathcal{O}(Y) \text{ injective}\end{array}\right\} \arrow[r, dash] & \left\{\begin{array}{c}\text{pairs } (N,Y) \text{ of lattices with}\\ Y \subseteq N \otimes H \text{ inducing an}\\ \text{isomorphism } Y \otimes H \cong N \otimes H\end{array}\right\}
\end{tikzcd}$$

The intuition behind this result is essentially the same as that of Mumford's seminal paper [Mum72], but expressed in a somewhat different language. Here I include a few paragraphs by way of translation; my notation follows that of [FC90].

Let K be a non-Archimedean field, T a split algebraic torus of rank n over K. We also fix a free $\mathbb{Z}$-module $Y \simeq \mathbb{Z}^n$ of the same rank. In [FC90], the authors distinguish three additional sets of data defining the completely degenerate Abelian variety $B_K := T/Y$:

i) a pairing $b : X^*(T) \otimes Y \to K^\times$;

ii) a homomorphism $\phi : Y \to X^*(T)$;

iii) a 'quadratic' function $a : Y \to K^\times$.

In more geometric terms, these data correspond to:

i) the 'period' lattice $b^\tau : Y \hookrightarrow T$;

ii) a cocycle ϕ for a θ-line bundle on T/Y;

iii) a metric on L.

We have already seen the lattice Y appearing in our $\mathbb{F}_1$-construction. Let us choose a section $H \to K^\times$ of the valuation, and suppose that b takes values in the image of this section. Then the analytic space B_K is obtained by base change from the $\mathbb{F}_1((t^{-H}))$-analytic space associated to an affine torus $B = (X_*(T) \otimes H)/Y$ along the induced map $\mathbb{F}_1((t^{-H})) \to K$, and we can make the identification $X^*(T) \cong \Lambda_{B/H}$.

I would like to argue that the θ-bundle is also defined at this level. Indeed, the very fact that the cocycle ϕ is takes values in the character group, rather than arbitrary Laurent polynomials, implies in particular that L is a monomial line bundle. More precisely, ϕ defines an element of

$$\mathrm{Pic}\left(B/\mathbb{F}_1((t^{-H}))\right) \quad \cong \quad H^1(B, \mathrm{Aff}(B,H)) \quad \cong \quad \mathrm{Hom}(Y,H) \oplus \mathrm{Hom}(Y, \Lambda_{B/H}),$$

where we use the origin of N to split the cotangent sequence $\mathrm{Aff}(N,H) \cong H \oplus \Lambda_{B/H}$. The fact that it even appears an element of the second factor $\mathrm{Hom}(Y,\Lambda_{B/H}) \cong \mathrm{Hom}(Y,X^*(T))$ implies that it is *metrisable*. I do not discuss the metric itself here.

I conclude this section by highlighting how the 'geometric' perspective of this paper stands up to the full generality of [FC90]:

- Using the technology presented here as written, it is only possible to get Abelian varieties with 'monomial structure constants', that is, such that Y acts on T with co-ordinates powers of the uniformiser. It is not difficult to reintroduce general $K^\times$ structure constants into the mix, but we defer that to a later project.
- The theta series cannot be defined directly over $\mathbb{F}_1((t))$ since they involve addition. In other words, the θ-bundles have no sections until they are pulled back to a base where addition is defined.
- One can also study families over a higher-dimensional base, but one will lose the interpretation in terms of affine manifolds.
- The affine manifolds perspective does not, it seems, offer any insight into the Abelian part of the Raynaud extension.

12.8 Rigid analytic spaces as twisted affine manifolds

One can also cook up an object that represents the overconvergent site of a purely analytic rigid space even without a trivialisation of the normal bundle to the punctures. The following discussion is informal.

Let X be paracompact, and pick a relatively normal model X^+. Then

$$\Sigma_{X^+}(H) = X\left(\mathbb{F}_1((t^{-H}))\right)$$

is a fan. Without the base field K to anchor it, the set of H-points has a scaling degree of freedom. The topological realisation $\Sigma_{X^+}(\mathbb{R})$ depends only on X, so we write simply $\Sigma_X(\mathbb{R})$.

As manifolds,

$$X^{\mathrm{sur}} \simeq \Sigma_{X^+}(\mathbb{R})/\mathbb{R}^\times_{>0}$$

so $\Sigma_X(\mathbb{R})$ can be considered as the bundle of positive rays in an *oriented real line bundle* $\ell^\vee$ on X^{sur}. The fibre of ℓ over any rational (resp. integral) point of X^{sur} comes with a set of rational (resp. integral) points respected by the structure group.

The locally constant sheaf $K_X^\times$ can no longer be considered as a subsheaf of $C^0(X^{\mathrm{sur}},\mathbb{R})$. Rather, it is a sheaf of functions

$$\ell^\vee \to \mathbb{R}$$

respecting the integral structures, that is, of *affine sections* of the dual line bundle ℓ. The zero section of ℓ is the global element $1 \in \Gamma K_X^\times$, and a non-vanishing global affine section of ℓ corresponds to a choice of structural morphism $X \to \mathrm{Spec}\,\mathbb{F}_1((t^{-\mathbb{Q}}))$.

12.28 Theorem. *Let X be a paracompact, boundaryless, purely analytic, locally Noetherian rigid analytic space over $\mathbb{F}_1$. Suppose that X is formally overconvergent over $\mathbb{F}_1$.*

Then X^{sur} is a manifold, and there is a canonical oriented real line bundle ℓ on X^{sur} and embedding $K_X^\times \hookrightarrow \Gamma(\ell/X)$ such that locally, these structures are isomorphic to those *on an affine manifold with $\ell = \mathbb{R}$ constant. Moreover, X can be recovered from the data $\ell/X^{\mathrm{sur}}, K_X^\times \hookrightarrow \Gamma(\ell/X)$.*

The line bundle ℓ, including its affine structure, computes the topological type of the normal bundle to the boundary in a log smooth model of X.

12.29 Example (Hopf map). Let S^3 be the punctured formal completion of the origin in $\mathbb{A}^2$. Identifying the blow-up of the plane at the origin with $\mathcal{O}_{\mathbb{P}^1}(-1)$, we obtain a 'Hopf fibration'

$$S^3 \to \mathbb{P}^1$$

the fibre over any k-point, with k an $\mathbb{F}_1$-field, being isomorphic to $\operatorname{Spec} k(\!(t)\!)$.

The punctured fan associated to the blow-up of $\hat{\mathbb{A}}^2$ is the lower quadrant in $\mathbb{R}^2$ divided into two cones by the negative diagonal, so $(S^3)^{\mathrm{sur}}$ is homeomorphic to a closed interval. Of course, both ℓ and $K_X^\times$ are topologically trivial. However, there is no *affine* trivialisation of ℓ: any element of $K_X^\times = \mathbb{Z}^2$ must annihilate some ray in $\mathbb{R}^2_{\leq 0}$ and hence, under the embedding $K_X^\times \hookrightarrow \ell$, vanish at the corresponding point of $(S^3)^{\mathrm{sur}}$.

In fact, a non-zero affine section of ℓ vanishes at exactly one point. This computes the Euler class, 1, of the conormal bundle $\mathcal{O}(1)$ to $\mathbb{P}^1$. I leave it to the reader to imagine his own generalisations.

13 De $\mathbb{F}_1(\!(t)\!)$ à $\mathbb{C}$

We already showed that there is a base change functor from the category of rigid analytic spaces over $\mathbb{F}_1(\!(t)\!)$ to any non-Archimedean field endowed with a topological nilpotent. Remarkably, one can also base change to *Archimedean* fields: there is a family of continuous monoid homomorphisms

$$\mathbb{F}_1(\!(t)\!) \to \mathbb{C}, \quad t \mapsto q$$

indexed by $q = e^\epsilon$ in the punctured open unit disc $\Delta^* \subset \mathbb{C}^\times$. The monoid $\mathbb{F}_1(\!(t)\!)$ does not distinguish between 'Archimedean' and 'non-Archimedean' topologies.

Although the formalisms are not immediately compatible, this can nonetheless be globalised to obtain a base change functor into the category of complex analytic spaces, in generalisation of the construction

$$B \mapsto TB/\epsilon\Lambda^\vee$$

of torus fibrations described in the introduction to part I.

Let S be a set with a $\mathbb{Z}$-indexed increasing filtration F, and define a norm

$$\|(c_f)_{f\in S}\| := \sum_{k\in\mathbb{Z}} \sum_{f\in F_kS\setminus F_{k-1}S} |q^k c_f|$$

on the $\mathbb{C}$-vector space $\bigoplus_S \mathbb{C}$. Its completion is the Banach space

$$\ell^1_q(S,F) := \left\{ (c_f)_{f\in S} \in \mathbb{C}^S \,\middle|\, \|(c_f)\| < \infty \right\}$$

of absolutely q-summable S-indexed sequences.

If A is a Banach $\mathbb{F}_1(\!(t)\!)$-algebra, then $A\hat{\otimes}_q\mathbb{C} := \ell^1_q(A, \operatorname{ord}_t)$ is a Banach $\mathbb{C}$-algebra with respect to the projective tensor product.

13.1 Lemma. *Suppose A is reduced and of finite type. Then $A\hat{\otimes}_q\mathbb{C}$ is the ring of holomorphic functions on* $\operatorname{Spec} A(\mathbb{C})$*.*

Proof. Let us choose a surjection $\mathbb{F}_1[\![t]\!]\{x_i\}_{i=1}^k \twoheadrightarrow A^+$. This embeds the affine algebraic variety $\operatorname{Spec} A^?(\mathbb{C})$ into an affine space $\mathbb{A}^k_\mathbb{C}$ in such a way that the t-adic norm on A^+ is the L^∞-norm on the unit polydisc of $\mathbb{A}^k_\mathbb{C}$. By the maximum principle, this is the same as the L^∞-norm on the unit torus. This is calculated by the formula for $\|-\|$ introduced above.

Since A is reduced, the discrete $\mathbb{C}$-algebra $A^? \otimes_q \mathbb{C}$ is exactly the set of polynomial functions on $\operatorname{Spec} A^?(\mathbb{C})$. Thus $A\hat{\otimes}_q\mathbb{C}$ is the completion of the space of polynomial functions for the L^∞-norm on the intersection $\operatorname{Spec} A(\mathbb{C})$ of $\operatorname{Spec} A^?(\mathbb{C})$ with the closed unit polydisc. Since this space is compact, the induced topology is the topology of uniform convergence, and hence the completion is the space of holomorphic functions. □

Suppose that $\operatorname{Spec} A \subseteq \operatorname{Spec} B \subseteq V$ are affine open immersions, and that $\operatorname{Spec} B$ is an overconvergent neighbourhood of $\operatorname{Spec} A/V$. By finiteness of global sections over blow-ups, $B\hat{\otimes}_q\mathbb{C} \to A\hat{\otimes}_q\mathbb{C}$ is a nuclear operator. It follows that any non-constant holomorphic function from $\operatorname{Spec} B(\mathbb{C})$ into the closed unit disc maps $\operatorname{Spec} A(\mathbb{C})$ into the open unit disc, that is, the latter is contained in the topological interior of the former.

If $U \subseteq \operatorname{Spec} A$ is an overconvergent open subset, then $U(\mathbb{C})$ is a complex analytic space without boundary. Indeed, by corollary 11.14, overconvergence means that every compact subset is in the topological interior of a larger compact subset.

The condition of local convexity (def. 11.18) being overconvergent-local, we can define the site $\mathbf{Rig}^{\mathrm{sur/lc}}_{\mathbb{F}_1(\!(t)\!)}$ of locally convex, overconvergent analytic spaces over $\mathbb{F}_1(\!(t)\!)$. It is generated by overconvergent spaces U admitting an open immersion into an affine object $\operatorname{Spec} A$.[19] To each such space, we have associated a complex analytic space $U(\mathbb{C})$. Following general principles, this extends to a spatial base change

$$(-) \times_q \operatorname{Spec}\mathbb{C} : \mathbf{Rig}^{\mathrm{sur/lc}}_{\mathbb{F}_1(\!(t)\!)} \to \mathbf{An}_\mathbb{C}$$

to the category of (possibly non-reduced) complex analytic spaces. One therefore has in general a map

$$\mu : X \times_q \operatorname{Spec}\mathbb{C} \to X^{\mathrm{sur}}$$

of spaces equipped with sheaves of topological monoids that specialises, in the case that X is boundaryless, to the construction of analytic torus fibrations described in the introduction.

A On the typing of points of rigid analytic spaces

Berkovich famously classified points on analytic curves C over a non-Archimedean field in to four ‘types’. With a small modification, it is easy to extend this to Huber’s framework and thus understand all points of the Riemann-Zariski space $\mathrm{RZ}(C)$ (§10.6). However, the generalisation to higher dimensions is somewhat more complicated.

For analytic spaces of finite type over $\mathbb{F}_1(\!(t)\!)$, the situation is sufficiently simple that we can give a complete solution. The pathological type IV points do not appear.

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[19] For $\mathbb{F}_1(\!(t^{-H})\!)$, one can show with a little effort that in fact all overconvergent spaces are locally convex, but we will not make that digression here.

A.1 Types I and III components

Let Δ be a rational polyhedron. We have constructed in §12.4 a continuous map

$$b : X_\Delta \to \Delta(\mathbb{R}_\infty)$$

where X_Δ is the rigid analytic space over $\mathbb{F}_1((t))$ associated to Δ, with a discontinuous section c.

Let $x \in X_\Delta$. The following invariants are visible at the level of its image bx:

A.1 Definitions. The *height*, or *type I codimension* or *component*, of x is the codimension of the (smallest) infinite face Δ_x of Δ containing x.

The *rational rank*, or *type III component*, is the dimension of the smallest rational subspace of Δ that contains x. Equivalently, it is the rank over $\mathbb{Q}$ of the linear map

$$\mathrm{Aff}(\Delta_x, \mathbb{R}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$$

induced by bx.

If the height, resp. rational rank of x equals the dimension of Δ, we say that x is *purely of type I* (resp. *III* or *irrational*). Note that Δ can have at most one purely type I point: its infinite vertex.

Intuitively, if x has type I (resp. III) component d_{I} (resp. d_{III}), it means that under a suitable co-ordinate system we may write

$$x = (-\infty, \cdots, -\infty | q_1, \cdots, q_\ell | r_1, \cdots, r_{d_{\mathrm{III}}})$$

with the first d_{I} co-ordinates equal to $-\infty$, the last d_{III} co-ordinates r_i relatively irrational, and each $q_i \in \sum_{j=1}^{d_{\mathrm{III}}} r_j \mathbb{Q}$.

A.2 Type II component

Let $x \in \Delta$. We may eliminate the types I and III components of x by replacing Δ with the quotient of Δ_x by the rational span of x; the image x_{II} of x in this quotient is called the *type II component* of x. If $x = x_{\mathrm{II}}$ we say it is or *purely of type II*. For describing the image bx in $\Delta(\mathbb{R}_\infty)$, we may also replace the morpheme 'type II' with *rational*.

In this section we will enumerate the additional data required to lift a point $y \in \Delta(\mathbb{R}_\infty)$ to Δ; since these data will depend only on the component y_{II}, it suffices to suppose that y is purely of type II.

A.2 Definition. Let $y \in \Delta(\mathbb{R})$ be purely of type II. An *oriented d_{II}-flag at y* is a complete flag $\rho_\bullet$ of half-spaces of the conormal cone $\Lambda^\vee_{\Delta,y}$ of dimensions up to $d_{\mathrm{II}} \in \mathbb{N}$; that is, ρ_1 is a ray (pointing into Δ) and ρ_{i+1} is a half-space with boundary the hyperplane $\pm\rho_i$.

It makes sense to ask if a jet in $N(\mathbb{R})$ at y - in the sense of differential geometry - lifts a specified flag ρ. A flag $\rho_\bullet$ is contained in a given polyhedron Δ if and only if there is a jet lifting $\rho_\bullet$ that points into $\Delta(\mathbb{R})$.

An oriented flag is equivalent to the data of a flag of half-spaces in the dual space $\Lambda_{\Delta,y}$ of *codimensions* up to d_{II}. Such a flag can be packaged in terms of the associated total order on $\Lambda/\rho^\perp_{d_{\mathrm{II}}}$ whose convex composition factors are $\pm\rho_i^\perp/\rho_{i+1}^\perp \cong \mathbb{Z}$ with the obvious ordering.

A.3 Proposition. *The set $\mathscr{J}_y\Delta$ of oriented flags of Δ at y is the set of surjective group homomorphisms*

$$v : \mathrm{Aff}_\Delta(N,\mathbb{Z}) \to \Gamma$$

into totally ordered groups Γ such that $v(F) \leq 0$ when $F \leq 0$ in a neighbourhood of y in $\Delta(\mathbb{R})$.

Thus oriented flags can be thought of as higher rank valuations of $\mathscr{O}\{\Delta\}$.

The set of oriented flags at y_{II}, where d_{II} is allowed to vary, carries a natural partial order by containment of flags. The unique zero-flag is the minimum element of $\mathscr{J}_y\Delta$. We will see that this partial order is precisely the specialisation order of $b^{-1}y$.

A.3 The set of points

The definition of the point-set topology of X_Δ is designed so that the conclusion of Stone's theorem holds. As such, its set of points may be identified with the set of *prime filters* of quasi-compact open subsets of X_Δ. We may identify this set in finitary terms.

A filter φ being prime means that if a set $U \in \varphi$ can be decomposed as the union $U_1 \cup U_2$ of two subsets, then either U_1 or $U_2 \in \varphi$. Since every quasi-compact open subset of X_Δ is the union of finitely many polyhedra, it follows:

A.4 Lemma. *Every prime filter of $\mathscr{U}^{\mathrm{qc}}_{/X_\Delta}$ is generated by polyhedra.*

Moreover, since one can check at the level of $\Delta(\mathbb{R}_\infty)$ whether a collection of polyhedra covers a given subset:

A.5 Lemma. *If $\varphi \in X_\Delta$, the filter of polyhedra containing φ is prime.*

We have therefore reduced the problem to describing the set of prime filters of polyhedra in the Hausdorff quotient $\Delta(\mathbb{R}_\infty)$.

Let me introduce the temporary notation $\mathscr{J}\Delta$ for the set of all pairs $(y,(y_{\mathrm{II}},\rho_\bullet))$ of a point $y \in \Delta(\mathbb{R}_\infty)$ and an oriented flag at the rational part y_{II}. The union of two sub-polyhedra contains such a pair if and only if one of the two constituents does. This produces a map from $\mathscr{J}\Delta$ to the set of prime filters of polyhedra.

A.6 Proposition. *The induced map $\mathscr{J}\Delta \to X_\Delta$ is bijective.*

Proof. This is the statement that every prime filter contains a unique oriented flag. Of course, every prime filter contains at least the length zero flag $\bigcap_{U\in\varphi} U$; the unicity reduces to the following elementary fact:

A.7 Lemma. *Elements of $\mathscr{J}\Delta$ are separated by polyhedra.*

In other words, for any two distinct flags we can find a pair of filters each containing just one of the points. □

A.8 Definition. Let $x \in X_\Delta$. The types I and III components $d_{\mathrm{I}}, d_{\mathrm{III}}$ of x are defined according to those of bx. The *type II codimension* or *residue height* of x is the length d_{II} of the flag $\rho_\bullet x \in \mathscr{J}_{bx}\Delta$ at bx_{II} corresponding to x under $X_\Delta \to \mathscr{J}\Delta$.

We say in this case that x is of type $(d_{\mathrm{I}}|d_{\mathrm{II}}|d_{\mathrm{III}})$.

A.4 Structure of the fibres of the Hausdorff quotient

Proposition A.6 identifies the closed subspace $b^{-1}y \subseteq \mathrm{sk}\Delta$, as a set, with $\mathscr{J}_y\Delta$. Moreover, the partial order of the set of filters by inclusion restricts, under this correspondence, to the partial order of weak jets by inclusion. In other words, this is also the specialisation order of $b^{-1}y$, as promised.

The irreducible closed subsets of $\mathscr{J}_y\Delta$ are therefore of the form

$$V_\rho := \{\rho' | \rho \subseteq \rho'\}$$

with $\rho \in \mathscr{J}_y\Delta$.

The embedding $b^{-1}y \hookrightarrow X_\Delta$ can be upgraded to an affine morphism of analytic spaces by equipping the left hand side with the pullback of the structure sheaf. With this structure, at least when y is purely of type II, $\mathscr{J}_y\Delta$ is the spectrum of the pair $(A;A^+)$ whose ring of integers is the $\mathbb{F}_1$-algebra associated to

$$\mathrm{Aff}^+_{\Delta,y}(N,\mathbb{Z}) := \{f \in \mathrm{Aff}_\Delta(N,\mathbb{Z}) | f \leq 0 \text{ in a neighbourhood of } y\}.$$

Note that this is not, of course, a Banach algebra topology.

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