A bivariant Yoneda lemma

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Abstract

Everyone knows that if you have a bivariant homology theory satisfying a base change formula, you get an action by correspondences. For theories in which the covariant and contravariant transfer maps are in mutual adjunction, these data are actually equivalent. In other words, a 2-category of correspondences is the universal way to attach right adjoints that satisfy a base change formula to a given 1-category.

Through a Yoneda philosophy of bivariant functors, I give a definition of correspondences in higher category theory and a proof of the preceding statement. The methods make no explicit reference to Segal spaces nor to any other model of (higher) 2-categories. As an application, we can give a neat construction of some symmetric monoidal structures.

Contents

1 Introduction

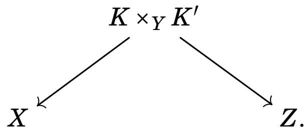
Let D be an ordinary category whose mapping objects are sets. We can define a 2-category $\mathrm{Corr}_{\mathrm{D}}$ of *correspondences in* D as follows:

- Objects of $\mathrm{Corr}_{\mathrm{D}}$ are objects of D.
- The mapping category $\mathrm{Corr}_{\mathrm{D}}(X,Y)$ is the category of spans

$$X \leftarrow K \rightarrow Y$$

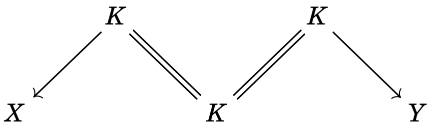
  and maps of spans over $X \times Y$.

- The composite of spans $X \leftarrow K \to Y$ and $Y \leftarrow K' \to Z$ is calculated by the fibre product



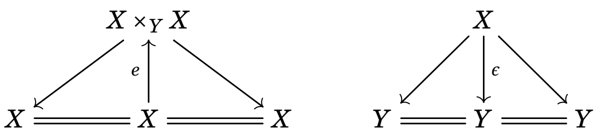
It can be shown 'by hand' that these sentences describe a 2-category $\mathrm{Corr}_{\mathrm{D}}$.

Any correspondence as above decomposes naturally as a composite



of a *wrong-way* or *contravariant* map $X \leftarrow K = K$ followed by a *right-way* or *covariant* map $K = K \to Y$. That is, the category of correspondences carries a natural *orthogonal factorisation system* $\mathrm{Corr}_{\mathrm{D}} = \langle \mathrm{D} \perp \mathrm{D}^{\mathrm{op}} \rangle$.

Furthermore, each wrong-way map $Y \stackrel{f}{\leftarrow} X$ is *right adjoint* to the right-way map $X \stackrel{f}{\to} Y$ obtained by considering it pointing the other way. We denote the right-way map by $f_!$ and the wrong-way map by f^*. The unit, respectively counit of this adjunction is depicted



Conversely, *any* adjoint pair of maps in $\mathrm{Corr}_{\mathrm{D}}$ is of this form. See [Hau14, Lemma 12.3].[1]

Finally, if $X \stackrel{g}{\to} Z \stackrel{f}{\leftarrow} Y$ are two maps in D, then both composites $f_! g^*$ and $g^* f_!$ are given by the same correspondence $X \leftarrow X \times_Y Z \to Y$, and hence these adjoints satisfy the *base change property* $f_! f g^* \cong g^* f_!$. Moreover, this is in a sense the *only* relation: to define a functor out of $\mathrm{Corr}_{\mathrm{D}}$ into some other 2-category, it is enough to define a covariant and a contravariant functor out of D such that this base change formula is satisfied.

Coherence and natural operations The above approach to correspondence categories involved defining the composition and checking associativity 'by hand'. Hence, studying correspondences in higher category theory presents two problems:

i) How do we define the associative structure of $\mathrm{Corr}_{\mathrm{D}}$ when D is a general $(\infty,1)$-category?

ii) When constructing functors from $\mathrm{Corr}_{\mathrm{D}}$ into a higher category, what compatibility do we require between the covariant and contravariant parts?

It's more or less obvious how one extends the fibre product structure to define a *Segal* $(\infty,1)$*-category* that models an $(\infty,2)$-category fitting the above specifications. This is the approach taken in [GR17, Part V], in which the main result of the present work also appears.[2]

In this paper, conversely, we adopt a more intrinsic approach based on 'natural operations' that exist in the internal logic of $(\infty,2)$-category theory. Consequently, the coherence of the *base change isomorphisms* (and other data) does not need to be checked explicitly. I have found it useful to develop a type theory style language for expressing this logic: the relevant terms are introduced in §2.2.

[1] In *op. cit.* the result is proved in an enlarged context (spans of spans, rather than only morphisms of spans, are allowed), but the preceding claims easily follow from the same constructions.

[2] Theorem 3.2.2 of *loc. cit.* is more general, considering functors that are bivariant only for a specified subclass of morphisms in the source category. This modification is also easily implemented in the approach of this paper.

Bivariant Yoneda philosophy A functor from an $(\infty,1)$-category into an $(\infty,2)$-category satisfying the above properties — one that takes every map in the source to one admitting a right adjoint, and the resulting adjoint pairs obey a base change formula — is called *bivariant*. Bivariant functors from D into the $(\infty,2)$-category of $(\infty,1)$-categories themselves form an $(\infty,2)$-category $\mathrm{Biv}_{\mathrm{D}}$, defined in §3.1. The main point of this paper is to construct a 'Yoneda API' for $\mathrm{Biv}_{\mathrm{D}}$.

To each object x of D, we assign a bivariant theory $\mathrm{Corr}_{\mathrm{D}}(x,-)$ of correspondences into x: this defines a faithful *Yoneda embedding* $\mathrm{D} \to \mathrm{Biv}_{\mathrm{D}}^{\mathrm{op}}$ which is itself bivariant. It is not full, but its essential image can be ascertained using the bivariant version of the *Yoneda lemma*:

Theorem (3.3.3)**.** *Let F be a bivariant functor of* D*. The natural evaluation mapping*

$$\mathrm{ev} : \mathrm{Biv}_{\mathrm{D}}[\mathrm{Corr}_{\mathrm{D}}(x,-), F(-)] \to F(x)$$

is an equivalence of categories, where $\mathrm{Biv}_{\mathrm{D}}[*,*]$ denotes bivariant natural transformations.

Sketch of proof. Considered via the Grothendieck construction as a bi-Cartesian fibration, the representable bivariant functor $\mathrm{Corr}_{\mathrm{D}}(x,-)$ is the free co-Cartesian fibration on the free Cartesian fibration on the singleton $\{x\} \to \mathrm{D}$. See §3.2, 3.3 for details. □

Hence, *defining* $\mathrm{Corr}_{\mathrm{D}} \subseteq \mathrm{Biv}_{\mathrm{D}}$ to be the *bivariant Yoneda image* of D — that is, the full $(\infty,2)$-subcategory spanned by representable bivariant functors — we obtain:

Corollary. *The Yoneda image is a* category of correspondences *in the sense that:*

- *For any $x, y : \mathrm{D}$, the category of maps between x and y in $\mathrm{Corr}_{\mathrm{D}}$ is $\mathrm{Corr}_{\mathrm{D}}(x,y)$.* (3.3.7)
- *under this identification, composition*

$$\mathrm{Corr}_{\mathrm{D}}^{\Delta^1} \times_{\mathrm{Corr}_{\mathrm{D}}} \mathrm{Corr}_{\mathrm{D}}^{\Delta^1} \to \mathrm{Corr}_{\mathrm{D}}^{\Delta^1}$$

  is isomorphic to the composite of correspondences computed by the fibre product. (3.4.1)

Finally, in §3.5 we take a more involved look at the functoriality of $\mathrm{Biv}_{\mathrm{D}}$ to deduce the promised universal property.

Theorem (3.5.1)**.** *The inclusion $\mathrm{D} \to \mathrm{Corr}_{\mathrm{D}}$ is universal among bivariant functors of* D*. That is, for any $(\infty,2)$-category* K*, restriction yields an equivalence of $(\infty,2)$-categories*

$$\mathrm{Biv}(\mathrm{D},\mathrm{K}) \cong 2\mathrm{Fun}(\mathrm{Corr}_{\mathrm{D}},\mathrm{K}).$$

This theorem provides the desired 'model-independent' way to produce functors out of correspondence categories: to be a functor of $\mathrm{Corr}_{\mathrm{D}}$ is simply a *property* of a functor of D.

Acknowledgements I thank Dennis Gaitsgory for taking an interest in this work, and whose questions drove me to think more carefully about the theoretical background upon which it is based. Thanks are also due to David Gepner, Aaron Mazel-Gee, and David Carchedi for helping me with the literature and sharing their perspectives on higher categories; I thank Rune Haugseng for drawing my attention to the work of Hinich on the enriched Yoneda lemma and for many educational discussions.

2 Higher category theory

The foundations of (∞,n)-category are currently in a substantially less settled form than for $(\infty,1)$-category theory.

The purpose of this section is to review the literature, especially for $n=2$, and to point out some gaps. Some, but not all, of the gaps are plugged here; the remaining foundational holes 2.1.7, 2.1.8 are at least somewhat precisely delineated (and are, in my opinion, so fundamental that no reader can reasonably doubt them).

2.1 Models of n-category theory

The underlying philosophy of this paper is to use only 'model-independent' constructions. However, strictly speaking, to make precise sense of this we would need a definition of the *syntax* of ∞-category theory, that is, the language and axioms thereof. At time of writing, no such axiomatisation has been formulated, and the best definition we have of ∞-categories is in terms of a model (quasi-categories). Hence, 'model-independence' for us is only a philosophical, rather than mathematical, criterion.

2.1.1 Aside. There is some hope that the intensional type theory discussed in [Uni13; RS17] can provide the axiomatisation needed to make sense of 'model-independence'. As it stands, that theory is not ready-to-use because it is not known to be equivalent to the ZFC set theory on which [HTT], and consequently this paper, is founded. However, I have no doubt that every statement in this paper is valid in homotopy type theory.

2.1.2 (1-categories)**.** This paper is based on the $(\infty,1)$-category theory of [HTT]. That is, $(\infty,1)$-categories are defined to be the fibrant objects in the Joyal model structure on simplicial sets. Using the results and constructions of *op. cit.* it is possible to completely avoid explicit reference to any simplicial set, and that is what we will do.

Since every category considered in this paper is 'higher', we will drop the prefix ∞ in the sequel and refer to quasi-categories simply as 1-categories.

2.1.3 (Approaches to n-categories)**.** There are several equivalent definitions of the homotopy theory of (∞,n)-categories (henceforth: n-categories).

- The original approach of C. Barwick [Bar05; GR17] defines them to be complete n-fold Segal spaces.
- As localisations of the category of presheaves on some predefined category of generators [Rez10; BSP11].
- Inductively as enriched categories [Sim11; GH13; Hin18].

Each of the above works define a 1-category of n-categories.

2.1.4 (Uniqueness of the theory of n-categories)**.** The results of [BSP11] provide a framework for comparing the 1-categories produced by the approaches enumerated in 2.1.3. It defines a theory of n-categories to be a presentable 1-category $n\mathbf{Cat}$ equipped with an embedding of a certain fixed category of generators $\mathbb{G}_n = \langle C_k\rangle_{k=0}^n$ consisting of 'n-globs' [BSP11, Def. 7.1]. Let us call the embedding $\mathbb{G}_n \hookrightarrow n\mathbf{Cat}$ the *orientation*.

The authors then compute [BSP11, §4] that

$$\mathrm{Aut}(n\mathbf{Cat}) = \mathrm{Aut}(\mathbb{G}_n) = (\mathbb{Z}/2\mathbb{Z})^n$$

with named generators op_k that 'reverse k-cells'. In particular, the space of *oriented* n-category theories is contractible.

2.1.5 Aside (Bergner-Rezk hierarchy). The use of strict n-categories in 2.1.3 can be avoided by basing the definitions on the Bergner-Rezk hierarchy $\{\Theta_n\}_{n:\mathbb{N}}$ [Ber07; Rez10], a variation that is allowed for in [BSP11, §11]. The $(1,1)$-categories Θ_n are defined inductively via a combinatorial procedure called the *categorical wreath product*. Their objects can be interpreted as strict n-categories obtained as a lattice-like concatenation of n-dimensional globs.

It is not hard to compute $\mathrm{Aut}(\Theta_n) = \mathrm{Aut}(\mathbb{G}_n)$ from the arguments of [BSP11, §4] and the fact that Θ_n is generated by $\mathbb{G}_n$.

2.1.6 (Orientations of the models)**.** The models listed in 2.1.3 are oriented as follows:

- An n-category theory produced by [BSP11, Thm. 11.2] comes equipped with an orientatation by (R.4) of *loc. cit*.

  Thus, Rezk Θ-spaces and n-fold complete Segal spaces are oriented by the construction of [BSP11, §13, 14];

- $(n-1)$**Cat**-enriched categories are oriented by transfer along the functor from the enriched model to the iterated Segal space model constructed in [Hau13, §7].

This fixes equivalences between the models and ensures that any combination thereof must commute.

2.1.7 (Further structures in category theory)**.** Category theory comprises more than just the concept of category and functor. Among the basic structures we will use in this paper are:

- n-categories are enriched in $(n-1)$-categories, that is, each pair of objects has a mapping $(n-1)$-category, and these $(n-1)$-categories have a composition law.
- Opposites act on these mapping $(n-1)$-categories, reversing composition.
- Given an n-category C and a specification of k-cells in C for $k = 0, \dots, n$, such that the source and target $(k-1)$-cell of each specified k-cell is in the specification, and stable under composition, there is a unique (replete) *subcategory* of C having exactly those spaces of k-cells and whose composition law is inherited from C.
- In particular, for $m \leq n$ there is a specification comprising all k-cell for $k \leq m$ and only invertible k-cells for $k > m$ whose associated subcategory is the *m-core* of C. This construction is a right adjoint functor to the inclusion $m\mathbf{Cat} \subseteq n\mathbf{Cat}$.

  These adjunctions are equivariant for the action of opposites.

Making the laws governing these structures precise would take us too far into the mechanics of enriched categories to be practical for this paper; hence, beyond the remarks presented here our approach will be to work under the — admittedly vague — assumption that they behave 'as expected'. More precise statements, with proofs in the enriched categories model, will appear in a companion paper [Mac19].

As far as I know, most current works that actually use n-categories use these structures on an ad hoc basis based on constructions in the iterated Segal space model. Some of these are made explicit in the case $n = 2$ in [GR17, §A.1].

2.1.8 ($(n+1)$-categories of n-categories)**.** Our intuitions tell us to expect that the collection of all n-categories form an $(n+1)$-category, not just an n-category. This intuition can be made precise by the enrichment by adjunctions technology of [Hin18, §6.3]. That is, $n\mathbf{Cat}$ is tensored over itself and hence attains an enrichment that extends the Cartesian closed structure. This enrichment has a universal property that I will not relate here.

The methods of *op. cit*. do not render the naturality of these enrichments completely transparent. However, it is likely the case that this construction defines a functor from the category of presentable $(n\mathbf{Cat}, \times)$-modules into $(n+1)\mathbf{Cat}$, but it is not clear to me that this follows straightforwardly from Hinich's theory. We use this hypothesis repeatedly to upgrade our constructions to 2-functors.

2.1.9 (Adjunctions)**.** There are, *a priori*, two possible definitions of n-adjunctions between n-functors $L : \mathrm{C} \rightleftarrows \mathrm{D} : R$ between n-categories:

i) via a unit-counit formulation, that is, as adjunctions in the 2-category of n-categories;

ii) as an n-natural equivalence

$$\mathrm{C}(-, R-) \cong \mathrm{D}(L-, -)$$

of bifunctors $\mathrm{C}^{\mathrm{op}} \times \mathrm{D} \to (n-1)\mathbf{Cat}$.

That these formulations are equivalent follows from exactly the same argument as in usual category theory (and, in fact, applies to any enriched categories). In particular, the implication ii)⇒i) uses the n-categorical Yoneda lemma [Hin18, Cor. 6.2.7]. We will use the formulation ii) for corollary 2.3.13.

2.2 Language of higher category theory

The language used in this paper is, so far as possible, *homotopy invariant*. This means that every statement — including every hypothesis and each intermediate step in every proof — is a construction of an ∞-functor of ∞-categories. These statements are generally made to be 'as functorial as possible' so that they not only respect the notion of equivalence in the domain of discourse — hence 'homotopy invariant' — but also some notion of morphism.

Non-homotopy invariant notions intervene only insofar as we use results from other works which, although their statements are homotopy-invariant, may be proved using non-invariant methods (such as those involving choosing a quasi-category model and playing with simplices, like many of the proofs in [HTT]).

2.2.1 (Notation — type theory)**.** This paper employs notational conventions from type theory, about which what little I know I learned from the IAS volume [Uni13] (especially appendix A.2). The formula templates outlined in this paragraph are sentences in a type-theoretic formal language whose full elaboration is beyond the scope of this paper.

Object declaration. Objects of categories (either variables or constants) are declared with a *colon*, i.e. $a:A$ states that a is an object of A (hence is equivalent to the usual notation $a \in A$). As usual, we may also write $f:A \to B$ or $g:C \cong D$ to declare a function or isomorphism; the category in which this morphism or isomorphism lives, if unclear, can be underset.

Functor declaration. We will make definitions via formulas of the form

$$\begin{aligned} &[\text{Generic } n\text{−functor}] \\ &a_0 : A_0, \ldots,\ a_k : A_k \quad \underset{n\mathbf{Cat}}{\vdash} \quad b : B \end{aligned} \tag{1}$$

where $n \in \{0,1,2\}$. Such a formula declares n-categories A_i, B, and an n-functor $\prod_0^k A_i \to B$ (the particular value of n specified by the underset to the turnstile); hence, b must be an n-functorial formula for an object of B in terms of the dummy variables a_i.

Currying. By definition, the functions of the symbols "$\vdash$" and "$\to$" are quite similar. The difference lies in what is made explicit: the first format makes all variables as well as the formula for the functor explicit, while the second provides a name for the functor but suppresses the variables. Passing between these notations is called 'currying': I use the format

$$a:A,\ b:B \quad \vdash \quad c:C \tag{2}$$

$$\equiv \quad a:A \quad \vdash \quad b.f:B \to C. \tag{3}$$

This reflects the Cartesian closure equivalence $n\mathrm{Fun}(A \times B, C) \cong n\mathrm{Fun}(A, n\mathrm{Fun}(B,C))$.

The optional prefix $b.$ shows that the variable that disappeared from the previous line is absorbed as an argument of f. A 'fully curried' version of the formula nFUN_GENERIC, with variable binding, reads

$$a_k.\ldots.a_1.f : \prod_{i=1}^{k} A_i \to B. \tag{4}$$

Dependent sums. It is also possible for each n-category A_i and B to depend n-functorially on the preceding ones, that is, (starting from $\iiint A_0 := A_0$):

- A_i is an anonymous n-functor $\iiint A_{i-1}\cdots A_0 \to n\mathbf{Cat}$.
- $\iiint A_i A_{i-1}\cdots A_0 := \int_{\iiint A_{i-1}\cdots A_0} A_i$, where $\int$ denotes the Grothendieck ('unstraightening') construction.

In this case, the formula declares a section of the projection $\iiint B A_k \cdots A_0 \to \iiint A_k \cdots A_0$. The Grothendieck construction needed to make sense of this is discussed in 2.3.5, and the first time I employ it for dependent categories is in 2.3.9. Currying in the form discussed above is not possible in the dependent case.

Natural transformation declaration. When an arrow or isomorphism symbol appears to the right of a turnstile, it is interpreted as a natural transformation of the variables to the left. Thus we write:

[Generic n-natural transformation]

$$a : A \underset{n\mathbf{Cat}}{\vdash} f : B \underset{D}{\to} C \tag{5}$$

[Generic n-natural isomorphism]

$$a : A \underset{n\mathbf{Cat}}{\vdash} f : B \underset{D}{\cong} C \tag{6}$$

for a natural transformation of functors $B(a) \to C(a)$ from A to D.

When the target category is $m\mathbf{Cat}$, we can make the natural transformation itself anonymous by nesting formulas, as in:

$$a : A \underset{n\mathbf{Cat}}{\vdash} (b : B \underset{m\mathbf{Cat}}{\vdash} c : C) \tag{7}$$

(brackets, added for clarification here, are not necessary). Provided that C does not depend on b, by naming the function this formula may again be curried into the format (5) to yield:

$$a : A \underset{n\mathbf{Cat}}{\vdash} b.f : B \underset{m\mathbf{Cat}}{\to} C. \tag{8}$$

2.2.2 Aside. Two advantages of writing in the format 2.2.1 are:

- It makes absolutely clear what level of functoriality in the arguments is expected, a detail which is easily obscured by writing in a less formal style.
- It allows us to manipulate anonymous functions, which is useful in arguments that use a lot of currying — such as those at the end of §3.5.

2.3 Grothendieck construction

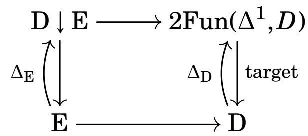
This section is about the 2-categorical *Grothendieck integral* $\int : 2\mathrm{Fun}(\mathrm{D}, 2\mathbf{Cat}) \cong 2\mathrm{Cart}_{\mathrm{D}}$. Some discussion of this construction is contained in [GR17, §A.2] to which I provide references. However, all statements in *loc. cit*. must be regarded as conditional on, at the very least, the list of 'unproved statements' found at the top of [GR17, §A.1].

Alternatively, the statements in 2.3.5 constitute a precise hypothesis. Grothendieck integration can be used to justify the following statements:

- Specification of subfunctors of functors into $2\mathbf{Cat}$ (assuming that specification of individual 2-subcategories makes sense, 2.1.7).

  This is used, for example, to define the 2-category of bivariant functors as a 2-functor of the domain and target 3.1.5.
- Extensionality of 2-natural isomorphisms. This statement is essentially our only way of constructing 2-equivalences, for example theorem 3.5.1.
- A suitably functorial version of the Yoneda lemma.

2.3.1 (Strict path category)**.** Let $p : \mathrm{D} \to \mathrm{E}$ be a 2-functor. The *(strict contravariant) path* or *comma category* is defined as a pullback



The strict path category commutes with taking 1-cores. Indeed, it follows from the adjointness property that $1\mathrm{Fun}(\Delta^1, \mathfrak{c}_1\mathrm{D}) = \mathfrak{c}_1 2\mathrm{Fun}(\Delta^1, \mathrm{D})$ and that $\mathfrak{c}_1$ preserves fibre products. Hence, comparing definitions, we find that the 1-core of the path category can be identified with the *slice* $\mathrm{D}_{/p}$ defined in [HTT].

When E is contractible, mapping to an object $d : \mathrm{D}$, I write also $\mathrm{D} \downarrow d$. In [HTT; GR17] this category would be denoted $\mathrm{D}_{/d}$.

2.3.2 (Lax path category)**.** Let $p : \mathrm{E} \to \mathrm{D}$ be a 2-functor. The *right lax right path category* of E over D can be constructed as a pullback

$$\begin{array}{ccc} \mathrm{D} \downarrow^{\ell} \mathrm{E} & \longrightarrow & \mathrm{RLax}(\Delta^1, \mathrm{D}) \\ {\scriptstyle \Delta_{\mathrm{E}}} \uparrow\downarrow & & {\scriptstyle \Delta_{\mathrm{D}}} \uparrow\downarrow {\scriptstyle \text{target}} \\ \mathrm{E} & \longrightarrow & \mathrm{D} \end{array}$$

where $\Delta^1 = \{0 \to 1\}$ is the usual 1-simplex category and RLax is the category of functors and right lax natural transformations [GR17, A.1, 3.2.7 and A.2, 5.1.1]. The section Δ_{D} is the identity section, and $\Delta_{\mathrm{E}} := (\mathrm{id}_{\mathrm{E}}, \Delta_{\mathrm{D}})$ in components. With this definition, it is clear that $\mathrm{D} \downarrow^{\ell} \mathrm{E}$ consistutes a 1-functor of E, and that Δ_{E} is a 1-natural transformation.

This 2-category has the following constituents:

- *Objects* are tuples $(e : E, d : D, \phi : d \to pe)$; I will also abbreviate such data as $(d \to e)$, thus suppressing p and the symbol ϕ from the notation.
- *Morphisms* $(d \xrightarrow{\phi} e) \to (d' \xrightarrow{\phi'} e')$ are lax commuting squares

$$\begin{array}{ccc} d & \longrightarrow & d' \\ \downarrow & \Rightarrow & \downarrow \\ pe & \longrightarrow & pe' \end{array}$$

  in D; more precisely, a member $\psi : \mathrm{D}(d, d')$ together with a functor from the square $[1,1]$ (see [GR17, A.1, 3.4] for notation) that sends the horizontal arrows to ϕ, ϕ' and the left-hand vertical arrow to $p(\psi)$.

We do not need to use any explicit description of the 2-cells.

In the case that D is a 1-category, the lax path category is a strict path category $\mathrm{D} \downarrow \mathrm{E}$.

2.3.3 (Cartesian fibrations)**.** The notion of (2-)Cartesian fibration of 2-categories and Cartesian transformations are defined in [GR17, A.2, Def. 1.1.2]. The category of Cartesian fibrations and Cartesian transformations over $\mathrm{D} : \mathbf{2Cat}$ forms a 1-full subcategory (A.1, 2.3.3 of *op. cit.*) $2\mathrm{Cart}_{\mathrm{D}}$ of the slice $\mathbf{2Cat} \downarrow \mathrm{D}$ of categories over D (A.2, 1.1.6; this category appears there decorated with the subscript 'strict').

(The definitions in [GR17] specialise to those provided in [HTT], hence the 1-core of our $1\mathrm{Cart}_{\mathrm{D}}$ is the category defined there.)

If $\mathrm{E} \to \mathrm{D}$ is a Cartesian fibration, then $\mathrm{D}^{\mathrm{op}_1} \to \mathrm{E}^{\mathrm{op}_1}$ is a co-Cartesian fibration. The equivalence

$$\mathrm{op} : \mathbf{2Cat} \downarrow \mathrm{D} \underset{\mathbf{1Cat}}{\cong} \mathbf{2Cat} \downarrow \mathrm{D}^{\mathrm{op}_1}$$

derived from the action of op_1 on $\mathbf{2Cat}$ and the 1-functoriality of path spaces restricts to a 1-equivalence

$$2\mathrm{Cart}_{\mathrm{D}} \underset{\mathbf{1Cat}}{\cong} 2\mathrm{coCart}_{\mathrm{D}^{\mathrm{op}}}.$$

In the case that D is a 1-category, we may bootstrap the 2-functorial op construction to obtain

$$1\mathrm{Cart}_{\mathrm{D}} \underset{\mathbf{2Cat}}{\cong} (1\mathrm{coCart}_{\mathrm{D}^{\mathrm{op}_1}})^{\mathrm{op}_2},$$

a formula which appears in 3.2.3, but is otherwise incidental to the logic.

2.3.4 (Maps out of the path category)**.** Lax comma categories have a universal mapping property as Cartesian fibrations: for any E, $\mathrm{D} \downarrow^{\ell} \mathrm{E}$ is (2-)Cartesian over D, and precomposition with Δ_{E} induces a 1-equivalence

$$2\mathrm{Cart}_{\mathrm{D}}(\mathrm{D} \downarrow^{\ell} \mathrm{E}, \mathrm{F}) \tilde{\to} 2\mathrm{Fun}_{\mathrm{D}}(\mathrm{E}, \mathrm{F})$$

for every $\mathrm{F} : \mathrm{Cart}_{\mathrm{D}}$. In the case that $\mathrm{E} = \mathbf{1}$ is a single object of D, a proof of this statement appears as [GR17, A.2, Prop. 5.3.2].

We will need to apply this in the case that E and D are 1-categories when we argue for the bivariant version in §3.3. This is established in [GHN15, Thm. 4.5] as an equivalence of spaces. The statement for mapping categories follows by consideration of $\mathrm{E} \times \Delta^1 \to \mathrm{D} \times \Delta^1$.

2.3.5 (Grothendieck integration)**.** The Grothendieck integral (a.k.a. lax colimit [GHN15, §7], a.k.a. 'unstraightening' [HTT, Chap. 2, 3]) of a functor $F : \mathrm{D}^{\mathrm{op}} \to \mathbf{2Cat}$ is denoted $\int_{\mathrm{D}} F$. It is defined in [GR17, Cor. 1.2.6(b)] as a 2-functor

$$\textstyle\int_{\mathrm{D}} : \mathbf{2Cat}^{\mathrm{D}^{\mathrm{op}}} \underset{\mathbf{2Cat}}{\to} \mathbf{2Cat} \downarrow \mathrm{D}.$$

The approach of [GR17] does not directly yield a usable formula for Grothendieck integration or its inverse. However, such a formula can be derived (corollary 2.3.10) from two properties of this equivalence:

- *Yoneda embedding*. The triangle

$$\begin{array}{ccc} \mathrm{D} & \xrightarrow{\ \mathrm{y}\ } & \mathbf{2Cat}^{\mathrm{D}^{\mathrm{op}}} \\ & {\scriptstyle \mathrm{D}\downarrow^{\ell}-}\searrow & \Vert \\ & & \mathrm{2Cart}_{\mathrm{D}} \end{array}$$

  is commutative, where y is the Yoneda embedding. In other words,

$$x : \mathrm{D} \quad \underset{\mathbf{2Cat}}{\vdash} \quad \textstyle\int_{\mathrm{D}} \mathrm{y}x \cong \mathrm{D} \downarrow^{\ell} x$$

  This is essentially the definition of y in [GR17, A.2, (5.6)].

- *Base change.* The Grothendieck equivalence is compatible with change of base: if $\mathrm{D} \to \mathrm{D}'$ is a functor, then the square

$$\begin{array}{ccc} \mathbf{2Cat}^{\mathrm{D}'} & \xrightarrow{\ \int_{\mathrm{D}'}\ } & \mathbf{2Cat} \downarrow \mathrm{D}' \\ {\scriptstyle \text{restriction}}\downarrow & & \downarrow{\scriptstyle \text{pullback}} \\ \mathbf{2Cat}^{\mathrm{D}} & \xrightarrow{\ \int_{\mathrm{D}}\ } & \mathbf{2Cat} \downarrow \mathrm{D} \end{array}$$

  commutes; in other words, the formation rule admits the strengthened form

$$\mathrm{D} : \mathbf{2Cat}^{\mathrm{op}} \quad \underset{\mathbf{2Cat}}{\vdash} \quad \textstyle\int_{\mathrm{D}} : \mathbf{2Cat}^{\mathrm{D}^{\mathrm{op}}} \to \mathbf{2Cat} \downarrow \mathrm{D}.$$

  This property is not completely transparent in [GR17]: see remark 2.3.6.

2.3.6 Aside (Status of the base change compatibility). The base change compatibility can be deduced from the definition of the 'straightening' 0-functor appearing in the proof of [GR17, A.2, Cor. 4.1.6]. (Actually, the cited proof defines the straightening functor in terms of a 'restriction map'

$$\mathbf{2Cat}_{/\mathbb{S}} \underset{\mathbf{0Cat}}{\to} \mathrm{PSh}(\Delta \times \Delta)\left(\mathrm{Sq}_{\bullet,\bullet}\mathbb{S}, \mathbf{2Cat}_{/[\bullet,\bullet]}\right)$$

which I could not find an explicit definition for; however, from the context it is clear that we are intended to use the formula

$$\mathbb{T}.(\phi : [m,n] \to \mathbb{S}).\phi^* \mathbb{T}$$

which surely sends pullbacks to restrictions.)

2.3.7 (Subfunctors)**.** Let $F : \mathrm{D} \to \mathbf{2Cat}$ be a 2-functor. Suppose that for each $x : \mathrm{D}$ we are given a 1-full subcategory $G(x) \subseteq F(x)$, such that for each $x \to y$, the image of $G(x) \to F(y)$ lies in $G(y)$. Then there is a unique 2-functor $G : \mathrm{D} \to \mathbf{2Cat}$ and 2-natural transformation $G \to F$ inducing the inclusion $G(x) \subseteq F(x)$ for each $x : \mathrm{D}$.

Indeed, this can be constructed via Grothendieck integration.

2.3.8 (Evaluation map)**.** By adjunction to the evaluation 2-functor

$$\mathrm{RLax}(\Delta^1, \mathrm{D}) * \Delta^1 \underset{2\mathbf{Cat}}{\to} \mathrm{D},$$

where $*$ denotes the Gray tensor product (A.1, 3), we obtain a map $\Delta^1 \to \mathrm{LLax}(\mathrm{RLax}(\Delta^1, \mathrm{D}), \mathrm{D})$ — that is, a left lax natural transformation from the source to the target map $\mathrm{RLax}(\Delta^1, \mathrm{D}) \to \mathrm{D}$.

When $\mathrm{D} = 2\mathbf{Cat}$, applying the Grothendieck construction yields a commuting triangle

$$\begin{array}{ccc} \int_{\mathrm{RLax}} s & \xrightarrow{\mathrm{ev}} & \int_{\mathrm{RLax}} t \\ & \searrow \quad \swarrow & \\ & \mathrm{RLax}(\Delta^1, 2\mathbf{Cat}) & \end{array}$$

by [GR17, A.2, Cor. 1.3.3(a)].

2.3.9 (Strong 2-Yoneda lemma)**.** The 2-Yoneda lemma, as we need it in this paper, goes as follows:

[Yoneda isomorphism]

$$x : \mathrm{D},\ F : 1\mathbf{Cat}^{\mathrm{D}} \underset{2\mathbf{Cat}}{\vdash} \mathrm{ev} : 1\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}(\mathrm{y}x, F) \underset{1\mathbf{Cat}}{\cong} F(x)$$

That is, the composite

$$\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}} \to \mathbf{Cat}^{(\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}})^{\mathrm{op}}} \to \mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}$$

of the Yoneda functor for $\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}$ with restriction along Yoneda for D is 2-equivalent to the identity.

This statement is established for *fixed* F and x in [Hin18, Prop. 6.2.7], with the map given by 'evaluation at id_x. However, the isomorphism is not shown to be 2-natural in F and x.

Lemma. *The composite*

$$\mathrm{Cart}_{\mathrm{D}} \to \mathrm{Cart}_{\mathrm{Cart}_{\mathrm{D}}} \to \mathrm{Cart}_{\mathrm{D}}$$

is equivalent to the identity.

Proof. By definition, this composite is the lax slice $\mathrm{D} \downarrow^{\ell}_{\mathrm{Cart}_{\mathrm{D}}} \mathrm{E}$; i.e. the pullback

$$\begin{array}{ccc} \mathscr{T} & \longrightarrow & \mathrm{Cart}_{\mathrm{D}} \downarrow^{\ell} \mathrm{E} \\ {\scriptstyle \mathrm{src}}\downarrow & & \downarrow \\ \mathrm{D} & \xrightarrow{\mathrm{D}\downarrow^{\ell} -} & \mathrm{Cart}_{\mathrm{D}} \end{array}$$

which we abbreviate as $\mathscr{T}$. Its objects are triangles

$$\begin{array}{ccc} \tilde{\mathrm{D}} & \xrightarrow{\mathrm{Cart}} & \mathrm{E} \\ & {\scriptstyle \text{ét}}\searrow \quad \swarrow & \\ & \mathrm{D} & \end{array}$$

for which $\tilde{\mathrm{D}} \to \mathrm{D}$ lies in the full (by the weak Yoneda lemma) subcategory $\mathrm{D} \subseteq 2\mathrm{Cart}_{\mathrm{D}}$ — that is, is some slice category $\mathrm{D} \downarrow^{\ell} x$ — and whose horizontal arrow is a Cartesian transformation. Morphisms are lax commuting triangles that commute (strictly) with projection to D.

Pulling back the evaluation map 2.3.8 along $\mathscr{T} \to \mathrm{Rax}(\Delta^1, 2\mathbf{Cat})$ yields a 2-functor

$$\textstyle\int_{\mathscr{T}} \mathrm{D} \downarrow^{\ell} \mathrm{src} = \mathscr{T} \times_{\mathrm{D}} \int_{x:\mathrm{D}} \mathrm{D} \downarrow^{\ell} x \to \mathrm{E}$$

over D.

My claim is that the full subcategory of $\int_{\mathscr{T}} \mathrm{src}$ spanned by the cone point of each fibre is equivalent to $\mathscr{T}$, and hence that the preceding formula defines a 2-functor $\mathscr{T} \to \mathrm{E}$. We then check that this is a map of Cartesian fibrations. □

2.3.10 Corollary. *The formulas for the functors in each direction are:*

$$F : 2\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}} \underset{2\mathbf{Cat}}{\vdash} \mathrm{D} \downarrow^{\ell} F : 2\mathrm{Cart}_{\mathrm{D}}$$
$$\mathrm{E} : 2\mathrm{Cart}_{\mathrm{D}} \underset{2\mathbf{Cat}}{\vdash} \mathrm{Cart}_{\mathrm{D}}(\mathrm{D} \downarrow^{\ell} -, \mathrm{E}) : 2\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}.$$

Grothendieck integration is the unique equivalence compatible with base change and the Yoneda embedding.

Proof. The diagram

$$\begin{array}{ccccc} \mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}} & \longrightarrow & \mathbf{Cat}^{(\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}})^{\mathrm{op}}} & \longrightarrow & \mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}} \\ & \searrow \;\circlearrowleft & \| & & \| \\ & & \mathrm{Cart}_{\mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}} & \longrightarrow & \mathrm{Cart}_{\mathrm{D}} \end{array}$$

is commutative, by the axioms for the Grothendieck integral 2.3.5.

By the strong Yoneda lemma, the top row is the identity. Hence, by commutativity (by definition for the left triangle and by base change compatibility for the right square) it will suffice to calculate the lower row. This is exactly the formula for $\int_{\mathrm{D}}$ provided.

Similarly, the other formula is simply the composite

$$\mathrm{Cart}_{\mathrm{D}} \to \mathbf{Cat}^{\mathrm{Cart}_{\mathrm{D}}^{\mathrm{op}}} \to \mathbf{Cat}^{\mathrm{D}^{\mathrm{op}}}.$$ □

The Grothendieck equivalence defined in [GR17] must therefore agree with that of [HTT].

2.3.11 Aside. I could not find a reference that directly states either the formulas 2.3.5 or the preceding form of the Yoneda lemma even for space-valued presheaves of an $(\infty,1)$-category. However, it can be deduced from the more complete results of [HTT, Chap. 5]; to be precise, that presheaves are generated under colimits by representables and that both the formulas define colimit-preserving functors.

2.3.12 Corollary (Extensionality of natural isos)**.** *Natural isomorphisms are extensional. That is, if $\eta : F \to G$ is a 2-natural transformation of 2-functors, and for each X in the source the induced map $\eta_X : FX \to GX$ is invertible, then η is a 2-natural isomorphism.*

Proof. By the Yoneda lemma, it will suffice to prove this for functors into 2**Cat**, hence for 2-Cartesian fibrations. Let $f : \mathrm{E} \to \mathrm{F}$ be a map of Cartesian fibrations that induces an equivalence on each fibre. Then f is an equivalence of 2-categories. □

2.3.13 Corollary. *Let* $\mathrm{D} : 1\mathbf{Cat}$. *The right comma category and inclusion functors fit into a 2-adjunction* $1\mathbf{Cat} \downarrow \mathrm{D} \rightleftarrows 1\mathrm{Cart}_{\mathrm{D}}$.

Proof. It follows from 2.3.4 that the functor

$$\mathrm{E} : 2\mathbf{Cat}_{\mathrm{D}} \;\vdash\; 2\mathrm{Fun}_{\mathrm{D}}(\mathrm{E}, -) : 1\mathbf{Cat}^{2\mathrm{Cart}_{\mathrm{D}}}$$

has image in representable functors. This, together with corollary 2.3.12, upgrades the isomorphism in 2.3.4 to a 2-natural isomorphism of bifunctors, hence a 2-adjunction. □

2.3.14. The corresponding story for a 2-adjunction

$$1\mathbf{Cat} \downarrow \mathrm{D} \leftrightarrows \mathrm{coCart}_{\mathrm{D}}$$

between categories over D and *covariant* functors $\mathrm{D} \to 1\mathbf{Cat}$ is completely analagous. We will need this 2-adjunction in §3.3.

3 Bivariance

3.1 Bivariant functors

In this section, D will be a 1-category that admits fibre products and K a 2-category.

3.1.1 (Conjugate mapping)**.** Let

$$\begin{array}{ccc} x_{00} & \xrightarrow{\bar{f}} & x_{01} \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ x_{10} & \xrightarrow{f} & x_{11} \end{array}$$

be a commuting square in a 2-category K (or a lax commuting square $[1,1] \to \mathrm{K}$ [GR17, A.1, 3.4]), and suppose that both f and $\tilde{f}$ admit a right adjoint $f^*, \tilde{f}^*$.

The 2-cell ϕ that exhibits the commutativity induces a *(right) conjugate mapping* (or, as the Australians call it, a *mate*) via a map (actually, an equivalence)

$$\mathrm{K}(x_{00}, x_{11})(g\tilde{f}, f\tilde{g}) \tilde{\to} \mathrm{K}(x_{01}, x_{10})(\bar{g}_! \bar{f}^*, f^* g_!).$$

This map is obtained by evaluating

$$\begin{aligned} \mathrm{K}(x_{00}, x_{11})(g_! \bar{f}_!, f_! \bar{g}_!) \times \mathrm{K}(x_{01}, x_{11})(g_! \bar{f}_! \bar{f}^*, g_!) \to & \mathrm{K}(x_{01}, x_{11})(f_! \bar{g}_! \bar{f}^*, g_!) \\ \cong & \mathrm{K}(x_{01}, x_{10})(\bar{g}_! \bar{f}^*, f^* g_!) \end{aligned}$$

at the image $f_!(\epsilon_g) : \mathrm{K}(x_{01}, x_{11})(g_! \bar{f}_! \bar{f}^*, g_!)$ of the counit ϵ_g of the $\bar{g}_! \dashv \bar{g}^*$ adjunction. Here in the second line I used the adjunction $\mathrm{K}(x_{10}, x_{11}) \leftrightarrows \mathrm{K}(x_{01}, x_{10})$ induced by postcomposition with $f_! \dashv f^*$.

3.1.2 Definition (Beck-Chevalley condition)**.** A commutative square is said to be *(right) Beck-Chevalley* if it admits a (right) conjugate and the conjugate mapping is an isomorphism.

Note that this condition is not invariant under flipping the square along its diagonal: it matters which arrows are said to be horizontal and which are vertical.

3.1.3 Definition (Bivariance)**.** A functor $H : \mathrm{D} \to \mathrm{K}$ is said to be a *bivariant functor*, or a (K-valued) *bivariant theory*, if:

- for each $f : x \to y$ in D, $f_! := H(f)$ admits a right adjoint. We denote this adjoint f^*;
- for each fibre square

$$\begin{array}{ccc} x \times_z y & \xrightarrow{f} & x \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow{f} & z \end{array}$$

  in D, the base change map $f_! g^* \to g^* f_!$ 3.1.1 is an equivalence in $\mathrm{K}(Hy, Hx)$.

Beck-Chevalley condition

If K is not specified, bivariant theories take values in 1**Cat**.

A natural transformation $\Phi : H_1 \to H_2$ is called a *morphism of bivariant theories* if it transforms right adjoints into right adjoints; more precisely, for each $f : x \to y$ in D the square

$$\begin{array}{ccc} H_1(x) & \xrightarrow{f_!} & H_1(y) \\ {\scriptstyle \Phi(x)}\downarrow & & \downarrow{\scriptstyle \Phi(x)} \\ H_2(x) & \xrightarrow{f_!} & \mathrm{H}_2(y) \end{array}$$

is a (right) Beck-Chevalley square.

3.1.4 (The 2-category of bivariant functors)**.** Transformations of bivariant functors $H_1 \to H_2$ of D into K are categorised as a full subcategory $\mathrm{Biv}(\mathrm{D},\mathrm{K})(H_1,H_2)$ of the usual category of natural transformations $2\mathrm{Fun}(\mathrm{D},\mathrm{K})(H_1,H_2)$. In this way, Biv(D,K) is a 1-full subcategory of the 2-category of functors Fun(D,K).

With these definitions, bivariant theories form a 1-fully embedded 2-subcategory Biv(D,K) of 2Fun(D,K). This category is stable under colimits and limits; in particular, it is 2-cocomplete (but as we do not need to use this fact, I do not provide a proof). In the case $\mathrm{K} = \mathbf{Cat}$, we abbreviate this to Biv_D.

3.1.5 (Composition)**.** Bivariant functors have the following naturality properties:

- if $G : \mathrm{K}_1 \to \mathrm{K}_2$ is a functor of 2-categories and $H : \mathrm{D} \to \mathrm{K}_1$ is a bivariant theory, then GH is bivariant; indeed, any functor of 2-categories preserves adjunctions.
- If $\phi : G_0 \to G_1$ is any map of 2-functors, then the induced morphism $G_0H \to G_1H$ is a bivariant natural transformation (the commutativity of the square is just a consequence of naturality).

Hence, any 2-natural formula 2.2.1 in $\beta : \mathrm{D}$, $H : \mathrm{Biv}(\mathrm{D},\mathrm{K})$ in which β appears prepended by H is bivariant in β.

Moreover, Biv is contravariant in D:

- If $F : \mathrm{D}_0 \to \mathrm{D}_1$ is a functor that preserves fibre products, and $H : \mathrm{D}_1 \to \mathrm{K}$ is a bivariant theory, then the composite HF is a bivariant theory.
- If $\psi : F_0 \to F_1$ is a map in $\mathrm{Fun}(\mathrm{D}_0,\mathrm{D}_1)$ which is *Cartesian* in the sense that for all $d_0, d_1 :$ D_0 the square

$$\begin{array}{ccc} F_0 d_0 & \longrightarrow & F_0 d_1 \\ \downarrow & & \downarrow \\ F_1 d_0 & \longrightarrow & F_1 d_1 \end{array}$$

  is Cartesian, then $HF_0 \to HF_1$ is a bivariant natural transformation.

Thus bivariant theories actually define a 2-bifunctor

$$\mathrm{Biv} : (1\mathbf{Cat}^{\mathrm{fp}})^{\mathrm{op}} \times 2\mathbf{Cat} \underset{2\mathbf{Cat}}{\longrightarrow} 2\mathbf{Cat}$$

defined on the 2-category $1\mathbf{Cat}^{\mathrm{fp}}$ of categories admitting fibre products, functors that preserve fibre products, and Cartesian natural transformations, which definition 3.1.3 cuts out as a 1-full subfunctor of the usual mapping space functor. This 2-naturality is used in the proofs of the 2-naturality properties of correspondence categories (corollaries 3.5.5, 3.5.7).[3] (There we will also use 2-Cartesian closure.)

3.1.6 (Syntax)**.** We expand the language of 2.2.1 to declare a functor as bivariant in a variable $a : A$ by postfixing ($\rightleftarrows$), so that, for example, we may write

$$a_1 : A_1\ (\rightleftarrows),\ a_2 : A_2 \quad \underset{2\mathbf{Cat}}{\vdash} \quad H(a_1, a_2) : B$$

when $H : A_1 \times A_2 \to B$ is a functor bivariant in a_1 but not necessarily in a_2. Note that context declaration is superfluous as regards a_1, because the source of a bivariant functor is by definition assumed to be a 1-category.

3.1.7 (Opposite notions)**.** Our definition of bivariant functor involved two choices of parity, reversed by op_1 and op_2. For this paragraph alone, let us give the objects defined in 3.1.3 the more precise name *right bivariant* functor. More generally, a functor $\mathrm{D} \to \mathrm{K}$ is said to be:

- *pseudo-bivariant* if it takes all morphisms in D to *left* adjoints in K;
- *pseudo-op-bivariant* if it takes all morphisms in D to *right* adjoints in K;

[3] Of course, a 3-categorical statement is also possible here, provided that we can also define 2-functor categories and cut out 1-full subcategories 3-naturally.

- (when D admits fibre products) *right (op-)bivariant* if it is pseudo-(op-)bivariant and satisfies the *base change condition*, i.e. it takes pullback squares to Beck-Chevalley squares;
- (when D admits amalgamated sums) *left (op-)bivariant* if it is pseudo-(op-)bivariant and satisfies the *collar change condition*, i.e. it takes pushout squares to Beck-Chevalley squares.

The 2-categories of pseudo-, right, and left bivariant functors and bivariant natural transformations are denoted psBiv, RBiv, LBiv, respectively. Their op-variants are indicated with a superscript 'o', i.e. $\mathrm{R^oBiv}$; naturally, in that case we ask that natural transformations preserve the *left* adjoints.

Taking opposites intertwines the various notions in the following way:

op_1 $\mathrm{psBiv(D,K) = ps^oBiv(D^{op}, K^{op_1})^{op_1}}$ and $\mathrm{RBiv(D,K) = L^oBiv(D^{op}, K^{op_1})^{op_1}}$ since passing to op_1 exchanges left with right adjoints and pullback squares with pushout squares.

op_2 $\mathrm{psBiv(D,K) = ps^oBiv(D, K^{op_2})^{op_2}}$ and $\mathrm{RBiv(D,K) = R^oBiv(D, K^{op_2})^{op_2}}$ because op_2 exchanges left and right adjoints.

This behaviour is summarised in the following table:

	-	op_1
-	RBiv	$\mathrm{L^oBiv}$
op_2	$\mathrm{R^oBiv}$	LBiv

3.2 Bi-Cartesian fibrations

For bivariant functors valued in **1Cat**, there is a theory of Grothendieck integration and sections analagous to the theory expounded in 2.3.5. This appears in [HA, §4.7.4], especially proposition 19.

3.2.1 (Bi-Cartesian fibrations)**.** Let $f : \mathrm{E} \to \Delta^1 = \{0 \xrightarrow{\delta} 1\}$ be a 1-functor. It is well-known that if every object of E(0) is the source of a (locally) co-Cartesian arrow, and every object of E(1) is the target of a (locally) Cartesian arrow, then the corresponding pullback and pushforward functors $\delta_! : \mathrm{E}(0) \rightleftarrows \mathrm{E}(1) : \delta^*$ are in adjunction with unit and counit fixed by the universal properties. Since I could not find a reference, I provide a proof:

Proof. Let $x : \mathrm{E}(1)$, and let $\delta^* x \to x$ be Cartesian over δ. Then by definition, $\delta^* x$ is final in the slice category $\mathrm{E}(0) \downarrow_{\mathrm{E}} x$, that is, the Grothendieck integral of the functor

$$y : \mathrm{E}(0) \underset{\mathbf{1Cat}}{\vdash} \mathrm{E}(y,x) : 0\mathbf{Cat}.$$

To put δ^* in adjunction with $\delta_!$, on the other hand, we must lift $\delta^* x$ (naturally in x) to an object of $\mathrm{E}(0) \downarrow_{\mathrm{E}(1)} x$, which is the integral of

$$y : \mathrm{E}(0) \underset{\mathbf{1Cat}}{\vdash} \mathrm{E}(1)(\delta_! y, x) : 0\mathbf{Cat}.$$

But by definition of co-Cartesian arrows,

$$y : \mathrm{E}(0)^{\mathrm{op}},\ x : \mathrm{E}(1) \underset{\mathbf{1Cat}}{\vdash} \mathrm{E}(y,x) \cong \mathrm{E}(1)(\delta_! y, x)$$

whence the result. □

Definition. A functor $\mathrm{E} \to \mathrm{D}$ is said to be *bi-Cartesian* if it is both Cartesian and co-Cartesian. A functor of bi-Cartesian fibrations over D is said to be a *bi-Cartesian transformation* if it respects Cartesian and co-Cartesian arrows. Hence, we define $\mathrm{biCart_D := Cart_D \cap coCart_D}$.

It follows that the Grothendieck equivalence 2.3.5 restricts to an equivalence of 2-categories

$$\mathrm{psBiv(D) \cong biCart_D},$$

natural in D. Hence, Grothendieck integration embeds the smaller categories of left and right bivariant theories fully faithfully into bi-Cartesian fibrations.

3.2.2 Example. When D admits fibre products, the target projection $\mathrm{D}^{\Delta^1} \to \mathrm{D}$ is a bi-Cartesian fibration. It is the covariant path category of the constant functor pt. Thus, taking sections yields a bivariant theory.

3.2.3 (Passage to adjoints)**.** In [GR17, §3] it is proved that from a pseudo-bivariant functor $H : \mathrm{D} \to \mathrm{K}$ can be constructed a pseudo-op-bivariant functor $\mathrm{D}^{\mathrm{op}} \to \mathrm{K}$ that sends each reversed arrow f in D^{op} to the right adjoint of $H(f)$. In the special case of functors satisfying the base (or collar) change property, this is a consequence of our main theorem 3.5.1 and the fact that $\mathrm{D}^{\mathrm{op}} \subseteq \mathrm{Corr}_{\mathrm{D}}$.

In the special case $\mathrm{K} = 1\mathbf{Cat}$ we are in a position to prove this already using the theory of bi-Cartesian fibrations 3.2.1. Indeed, taking opposites defines a 2-equivalence

$$\mathrm{biCart}_{\mathrm{D}} \cong \mathrm{biCart}_{\mathrm{D}^{\mathrm{op}}}^{\mathrm{op}_2}$$

which, in the case $\mathrm{D} = \mathbf{1}$ is contractible, specialises to the usual opposite equivalence $1\mathbf{Cat} \cong 1\mathbf{Cat}^{\mathrm{op}_2}$. It follows that

$$\begin{aligned}
\mathrm{psBiv}(\mathrm{D}) &= \mathrm{biCart}_{\mathrm{D}} && \text{by Grothendieck integration} \\
&= \mathrm{biCart}_{\mathrm{D}^{\mathrm{op}}}^{\mathrm{op}_2} && \text{by opposites} \\
&= \mathrm{psBiv}(\mathrm{D}^{\mathrm{op}})^{\mathrm{op}_2} && \text{by taking sections} \\
&= \mathrm{ps}^{\mathrm{o}}\mathrm{Biv}(\mathrm{D}^{\mathrm{op}}, \mathbf{Cat}^{\mathrm{op}_2}) && \text{by 3.2.1} \\
&= \mathrm{ps}^{\mathrm{o}}\mathrm{Biv}(\mathrm{D}^{\mathrm{op}}) && \text{by opposites in } \mathbf{Cat}
\end{aligned}$$

and that this equivalence exchanges $\mathrm{RBiv}_{\mathrm{D}}$ with $\mathrm{LBiv}_{\mathrm{D}^{\mathrm{op}}}^{\mathrm{op}_2} \cong \mathrm{L}^{\mathrm{o}}\mathrm{Biv}_{\mathrm{D}^{\mathrm{op}}}$. This argument appears in §4.1 of *loc. cit.*.

3.2.4 Aside. Just as right bivariant theories are corepresented by a 2-category $\mathrm{Corr}_{\mathrm{D}}$ whose mapping objects are categories of spans (theorem 3.5.1), *left* bivariant theories are corepresented by a 2-category $\mathrm{Bord}_{\mathrm{D}}$ of 'bordisms' whose mapping objects are categories of *cospans*.

The choice of the adjectives 'left' and 'right' in 3.1.7 reflects the fact that their corepresenting objects $\mathrm{Bord}_{\mathrm{D}}$, resp. $\mathrm{Corr}_{\mathrm{D}}$, admit a kind of calculus of left, resp. right fractions, while the equivalence in the last line of 3.2.3 reflects an equivalence $\mathrm{Bord}_{\mathrm{D}} \cong \mathrm{Corr}_{\mathrm{D}^{\mathrm{op}}}^{\mathrm{op}_2}$.

The purpose of this section is to construct a commutative square

$$\begin{array}{ccc}
\mathrm{Cart}_{\mathrm{D}} & \rightleftarrows & \mathrm{biCart}_{\mathrm{D}} \\
\downarrow & & \downarrow \\
\mathbf{Cat} \downarrow \mathrm{D} & \rightleftarrows & \mathrm{coCart}_{\mathrm{D}}
\end{array}$$

by showing that the constructions of the covariant path category and relative left Kan extension on the bottom line, as discussed in 2.3.4, restrict to the top line. Moreover, we will see that the contravariant path category on a co-Cartesian fibration satisfies the base change property, hence its functor of sections is right bivariant.

3.2.5 (Objects)**.** Suppose D admits fibre products. Let $\pi : \mathrm{E} \to \mathrm{D}$ be a Cartesian fibration. Then $\mathrm{E} \downarrow \mathrm{D} \to \mathrm{D}$ is bivariant:

- for $f : \alpha \to \beta$ in D, the pullback is defined by asking that the squares

$$\begin{array}{ccc}
f^*x & \longrightarrow & x \\
| & & | \\
\alpha \times_\beta \pi x & \longrightarrow & \pi x \\
\downarrow & & \downarrow \\
\alpha & \xrightarrow{f} & \beta
\end{array}$$

  be Cartesian.

- Base change follows from the isomorphism

$$x \times_\alpha \alpha \times_\gamma \beta \cong x \times_\gamma \beta$$

which is a consequence of the definition of pullback and associativity of fibre products.

3.2.6 (Morphisms)**.** Let $\mathrm{E} \to \mathrm{D}$ be Cartesian, $\mathrm{F} \to \mathrm{D}$ bi-Cartesian and having the base change property. We will exhibit a commuting square

$$\begin{array}{ccc} \mathbf{Cat}_{\mathrm{D}}(\mathrm{E},\mathrm{F}) & = & \mathrm{coCart}_{\mathrm{D}}(\mathrm{E} \downarrow \mathrm{D}, \mathrm{F}) \\ \uparrow & & \uparrow \\ \mathrm{Cart}_{\mathrm{D}}(\mathrm{E},\mathrm{F}) & = & \mathrm{biCart}_{\mathrm{D}}(\mathrm{E} \downarrow \mathrm{D}, \mathrm{F}) \end{array}$$

by showing that the relative left Kan extension 3.3 of a map $\phi : \mathrm{E} \to \mathrm{F}$ of Cartesian fibrations itself preserves Cartesian arrows.

Proof. Denote the extension of ϕ by $\tilde{\phi}$. Consider a Cartesian arrow

$$\begin{array}{ccc} x \times_\beta \alpha & \xrightarrow{g} & x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \alpha & \xrightarrow{g} & \beta \end{array}$$

in $\mathrm{E} \downarrow \mathrm{D}$. Note that $(x \to \beta) = f_!(x \to p_{\mathrm{E}} x)$ and $(g^* x \to \alpha) = \bar{f}_!(g^* x \to p_{\mathrm{E}} g^* x)$.

Then

$$\begin{aligned} \tilde{\phi}(x \times_\beta \alpha \to \alpha) &= \bar{f}_! \phi(\bar{g}^*(x \to p_{\mathrm{E}} x)) && \tilde{\phi} \text{ co-Cartesian} \\ &= \bar{f}_! \bar{g}^* \phi(x \to p_{\mathrm{E}} x) && \phi \text{ Cartesian} \\ &= g^* f_! \phi(x \to p_{\mathrm{E}} x) && \text{base change in F} \\ &= g^* \tilde{\phi}(x \to p_{\mathrm{E}} x) && \tilde{\phi} \text{ co-Cartesian} \end{aligned}$$

where we didnt actually need the formula at all. □

3.2.7 Proposition (Bivariant path category)**.** *Right Yoneda expansion and restriction respect the embeddings of co- and bi-Cartesian fibrations, hence define a left 2-adjoint*

$$- \downarrow \mathrm{D} : \mathrm{Cart}_{\mathrm{D}} \underset{\mathbf{2Cat}}{\to} \mathrm{RBiv}(\mathrm{D})$$

to the inclusion functor.

By composition we obtain

$$(\mathrm{D} \downarrow -) \downarrow \mathrm{D} : \mathbf{Cat} \downarrow \mathrm{D} \underset{\mathbf{2Cat}}{\to} \mathrm{RBiv}(\mathrm{D})$$

where the covariant path category is defined using the source projection of $(\mathrm{D} \downarrow -)$.

Proof. The 2-adjunction between $1\mathbf{Cat} \downarrow \mathrm{D}$ and $\mathrm{coCart}_{\mathrm{D}}$ was established in 2.3.4. It follows also from 3.2.6 that the counit is a map of (right) bivariant functors. □

3.2.8 Aside (Left bivariant version). Corresponding results are true in the left bivariant case: the *contravariant* path category on a co-Cartesian fibration is left bivariant, and maps into a left bivariant theory may be freely extended.

3.3 Bivariant Yoneda lemma

3.3.1 (Correspondences)**.** Denote by Λ the free category on the picture

$$\begin{array}{ccccc} & & s_{01} & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ s_0 & & & & s_1 \end{array}$$

Its unique symmetry is denoted σ. A *correspondence* or *span* in D is a functor $\Lambda \to \mathrm{D}$. We will sometimes abbreviate the data of an object of D^Λ by (p,q), representing the left and right arrows.

By construction, the category of spans in D fits into a pullback square

$$\begin{array}{ccc} \mathrm{D}^\Lambda & \xrightarrow{p} & \mathrm{D}^{\Delta^1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{src}} \\ \mathrm{D}^{\Delta^1} & \xrightarrow{\mathrm{src}} & \mathrm{D} \end{array}$$

from which it follows that

$$\mathrm{E} : \mathbf{Cat} \downarrow \mathrm{D} \quad \vdash \quad \mathrm{E} \times_{\mathrm{D}} \mathrm{D}^\Lambda \cong (\mathrm{D} \downarrow \mathrm{E}) \downarrow \mathrm{D}$$

is the free co-Cartesian fibration on the free Cartesian fibration — that is, the free bivariant theory — on $\mathrm{E} \to \mathrm{D}$. (By convention, the fibre product in this formula uses the first projection $\mathrm{ev}(s_0)$ of D^Λ.)

This is the bivariant path category of proposition 3.2.7. In particuar, the projection $\mathrm{ev}_1 : \mathrm{D}^\Lambda \to \mathrm{D}$ is the free bi-Cartesian fibration on the identity functor of D.

The two outer projections are exchanged by the flip Λ. The fibre over (x,y) of the outer projections is denoted $\mathrm{Corr}_{\mathrm{D}}(x,y) : 1\mathbf{Cat}$.

3.3.2 (Yoneda embedding)**.** In particular, the left projection $\mathrm{D}^\Lambda \to \mathrm{D}$ being a Cartesian fibration, it determines functors

$$\mathrm{D}^{\mathrm{op}} \to \mathrm{RBiv}(\mathrm{D}) \subseteq \mathrm{biCart}_{\mathrm{D}}$$

$$\mathrm{D} \to \mathrm{RBiv}(\mathrm{D})$$

(see 3.2.3). The former is right bivariant, while the latter is left op-bivariant. Via the flip, these functors have the same essential image, but not the same morphisms.

We will prioritise the right bivariant functor

$$\mathrm{y} : \mathrm{D} \to \mathrm{Biv}(\mathrm{D})^{\mathrm{op}}$$

obtained from the contravariant version 3.1.7. This is the *bivariant Yoneda embedding*. We will see that unlike the usual case, it is faithful but not full 3.3.7.

This Yoneda functor associates to an object $x : \mathrm{D}$ the bivarant functor $y.\mathrm{Corr}_{\mathrm{D}}(x,y)$, and to a morphism $x \to x'$ it associates the pullback transformation $\mathrm{Corr}_{\mathrm{D}}(x',-) \to \mathrm{Corr}_{\mathrm{D}}(x,-)$.

3.3.3 Corollary (Bivariant Yoneda lemma)**.** *Evaluation at the identity correspondence*

$$x : \mathrm{D}\ (\rightleftarrows),\ H : \mathrm{Biv}_{\mathrm{D}} \underset{\mathbf{2Cat}}{\vdash} \epsilon : \mathrm{Biv}_{\mathrm{D}}(\mathrm{y}x, H) \to H(x)$$

is an equivalence.

Proof. The sequence of adjunction isomorphisms

$$\begin{aligned} x : \mathrm{D},\ H : \mathrm{Biv}_{\mathrm{D}} \underset{\mathbf{2Cat}}{\vdash} \mathrm{Biv}_{\mathrm{D}}(\mathrm{y}x, H) &\cong \mathrm{Cart}_{\mathrm{D}}\left(\mathrm{D} \downarrow x, \textstyle\int_{\mathrm{D}} H\right) && \text{(proposition 3.2.7)} \\ &\cong H(x) && \text{(2-Yoneda lemma).} \end{aligned}$$

□

give the natural isomorphism at the level of 2-functors of x, H. Since the right-hand side is also bivariant in $x : \mathrm{D}$, it is even an isomorphism of bivariant functors.

3.3.4 (Induced map on correspondences: local version)**.** Applying the Yoneda isomorphism of the bivariant functor $y.\mathrm{K}(Hx, Hy)$

$$\mathrm{K}(Hx, Hx) \cong \mathrm{Biv}_{\mathrm{D}}[\mathrm{Corr}_{\mathrm{D}}(x,-), \mathrm{K}(Hx, H-)]$$

to the identity of Hx yields a *local action of correspondences*

$$y : \mathrm{D}\,(\rightleftarrows) \quad \vdash \quad (p,q) : \mathrm{Corr}_{\mathrm{D}}(x,y) \quad \vdash \quad H(p,q) : \mathrm{K}(Hx, Hy)$$

in particular, a (1-)functor $\mathrm{Corr}_{\mathrm{D}}(x,y) \to \mathrm{K}(Hx,Hy)$ for fixed x, y.

3.3.5 Definition (Correspondences)**.** Let D be a 1-category. A bivariant functor $\mathrm{D} \to \mathrm{K}$ is said to exhibit K as a *2-category of correspondences of* D if it is essentially surjective and induces, via 3.3.4, an equivalence of categories

$$\mathrm{Corr}_{\mathrm{D}}(x,y) \tilde{\to} \mathrm{K}(Hx, Hy)$$

for each $x, y : \mathrm{D}$. If $\mathrm{Corr}_{\mathrm{D}} : 2\mathbf{Cat}$ is a category of correspondences for D, then the notation $\mathrm{Corr}_{\mathrm{D}}(x,y)$ may unambiguously be interpreted either following 3.3.7 or as the category of morphisms from x to y in $\mathrm{Corr}_{\mathrm{D}}$. We will see in §3.5 that $\mathrm{Corr}_{\mathrm{D}}$ itself is uniquely determined.

3.3.6 Definition (Representable bivariant functors)**.** A bivariant functor $\mathrm{D} \to 1\mathbf{Cat}$ is said to be *representable* if it is isomorphic to one of the form $\mathrm{Corr}_{\mathrm{D}}(x,-)$. The *Yoneda image* of D is the full 2-subcategory of $\mathrm{RBiv}_{\mathrm{D}}$ spanned by the representable (right) bivariant functors.

3.3.7 Corollary (Mapping category)**.** *The Yoneda image of* D *is a 2-category of correspondences.*

Proof. Apply corollary 3.3.3 to the theory $H = \mathrm{Corr}_{\mathrm{D}}(-,y)$ for various y. □

3.3.8 Aside (Left bivariant version). Dually, if D is a category admitting pushouts, then we define the *bordism* category $\mathrm{Bord}_{\mathrm{D}} \subseteq \mathrm{LBiv}_{\mathrm{D}}$ to be the full 2-subcategory spanned by representable *left* bivariant functors.

3.4 Composition of correspondences

The purpose of this section is to relate composition of correspondences by taking fibre product of the kernels with composition in the image of a bivariant functor.

3.4.1 Proposition (Composition)**.** *This local action intertwines composition of correspondences by fibre products with composition in* K*, i.e. the diagram*

$$\begin{array}{ccc} \mathrm{Corr}_{\mathrm{D}}(x,y) \times \mathrm{Corr}_{\mathrm{D}}(y,z) & \xrightarrow{f.p.} & \mathrm{Corr}_{\mathrm{D}}(x,z) \\ \downarrow & & \downarrow \\ \mathrm{K}(Hx,Hy) \times \mathrm{K}(Hy,Hz) & \xrightarrow{\circ} & \mathrm{K}(Hx,Hz) \end{array}$$

is commutative.

Proof. By combining lemmas 3.4.4 and 3.4.3, below. □

3.4.2 (Action mapping)**.** Let $H : \mathrm{Biv}(\mathrm{D})$. The free functor $\mathrm{Corr}_{\mathrm{D}} : \mathbf{Cat} \downarrow \mathrm{D} \to \mathrm{biCart}_{\mathrm{D}}$ commutes with product by a fixed category. Hence, for fixed $y : \mathrm{D}$ the inclusion $Hy \subseteq \int_{z:\mathrm{D}} Hz$ of a fibre induces, via proposition 3.2.7 an equivalence

$$\mathrm{Biv}_{z:\mathrm{D}}(\mathrm{Corr}_{\mathrm{D}}(y,z) \times Hy, Hz) \cong \mathrm{Fun}(Hy, Hy).$$

Applying this to the identity functor of Hy, we obtain an action mapping

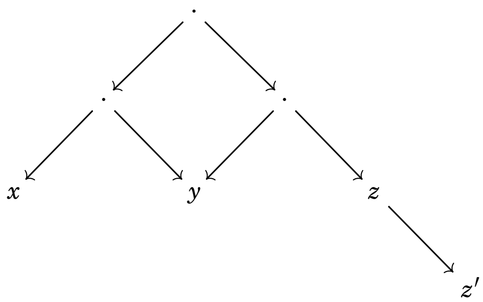
$$\mathrm{act}_{\mathrm{D},H,y} : \mathrm{Corr}_{\mathrm{D}}(y,-) \times Hy \to \int_{z:\mathrm{D}} Hz$$

by push-pull. Here I used the free property of $\mathrm{Corr}_{\mathrm{D}}(y)$ and the fact that being free commutes with Cartesian product by a fixed category (in this case Hy).

3.4.3 Lemma (Action of correspondences on themselves)**.** *In the case* $H = \mathrm{Corr}_{\mathrm{D}}(-,z)$*, this action is given by the* composition of correspondences $\mathrm{Corr}_{\mathrm{D}}(y,-)\times\mathrm{Corr}_{\mathrm{D}}(x,y)\to\mathrm{Corr}_{\mathrm{D}}(x,-)$.

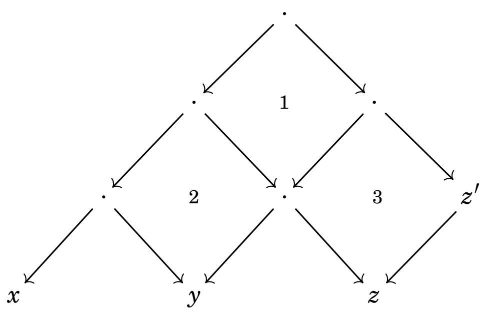
Proof. The restriction to $\{\mathrm{id}_y\}\times\mathrm{Corr}_{\mathrm{D}}(x,y)\to\mathrm{Corr}_{\mathrm{D}}(x)$ is simply the inclusion of the fibre. Hence, by the uniqueness principle of the action map, it suffices to check that this is a map of bi-Cartesian fibrations.

- Let $z\to z'$. The output is the roof of the diagram



regardless of the ordering of [composition] and [pushforward].

- Let $z'\to z$. The output is the roof of



which calculates [composition then pullback] because square 3 and $2+1$ are Cartesian and [pullback then composition] because square 2 and $3+1$ are Cartesian.

□

3.4.4 Lemma (Action intertwines with composition)**.** *The diagram*

$$\begin{array}{ccc} \mathrm{Corr}_{\mathrm{D}}(y,z)\times\mathrm{K}(Hx,Hy) & & \\ {\scriptstyle (H,\mathrm{id})}\downarrow & \searrow^{\mathrm{act}} & \\ \mathrm{K}(Hy,Hz)\times\mathrm{K}(Hx,Hy) & \xrightarrow{\circ} & \mathrm{K}(Hx,Hz) \end{array}$$

in which the upper arrow is the local action by correspondences on the functor $\mathrm{K}(H-,Hz)$ *and the lower is composition in* K*, commutes.*

Proof. Fixing y, z but allowing x to vary, we apply the commutativity of the square

$$\begin{array}{ccc} \mathrm{Fun}[\mathrm{K}(y,z),\mathrm{K}(y,y)\times\mathrm{K}(y,z)] & = & \mathrm{Biv}_{x:\mathrm{D}}[\mathrm{Corr}_{\mathrm{D}}(x,y)\times\mathrm{K}(y,z),\mathrm{K}(x,y)\times\mathrm{K}(y,z)] \\ \downarrow & & \downarrow \\ \mathrm{Fun}[\ \mathrm{K}(y,z)\ ,\ \mathrm{K}(y,z)\ ] & = & \mathrm{Biv}_{x:\mathrm{D}}[\ \mathrm{Corr}_{\mathrm{D}}(x,y)\times\mathrm{K}(y,z)\ ,\ \mathrm{K}(x,z)\ ] \end{array}$$

where the vertical arrows are induced by composition $x.\mathrm{K}(Hx,Hy)\times\mathrm{K}(Hy,Hz)\to\mathrm{K}(Hx,Hz)$ and the horizontal equalities by the 2-adjunction of proposition 3.2.7.

We apply to the composition map $\mathrm{K}(Hx,Hy)\times\mathrm{K}(Hy,Hz)\to\mathrm{K}(Hx,Hz)$, considered as a map of functors of x, the 2-adjunction isomorphism (prop. 3.2.7)

$$\mathrm{Fun}[\mathrm{K}(Hy,Hz),-]\cong\mathrm{Biv}_{x:\mathrm{D}}[\mathrm{Corr}_{\mathrm{D}}(x,y)\times\mathrm{K}(Hy,Hz),-].$$

The action map is by definition transferred over from the identity of $K(Hy, Hz)$. Meanwhile H is transferred rom the identity map of Hy. □

3.5 Universal property of correspondences

The purpose of this section is to prove the main result of this paper, a 'global' version of corollary 3.3.7:

3.5.1. Denote by $\mathrm{Corr}_\mathrm{D} \subset \mathrm{Biv}_\mathrm{D}^{\mathrm{op}}$ the full subcategory spanned by representable objects. The inclusion $\mathrm{D} \subseteq \mathrm{Corr}_\mathrm{D}$ induces, by 3.1.5, a functor of *restriction* $2\mathrm{Fun}(\mathrm{Corr}_\mathrm{D}, \mathrm{K}) \to \mathrm{Biv}(\mathrm{D}, \mathrm{K})$.

Theorem. *Restriction along the inclusion* $\mathrm{D} \subseteq \mathrm{Corr}_\mathrm{D}$ *induces a 2-equivalence*

$$2\mathrm{Fun}(\mathrm{Corr}_\mathrm{D}, \mathrm{K}) \underset{\mathbf{2Cat}}{\xrightarrow{\sim}} \mathrm{Biv}(\mathrm{D}, \mathrm{K}).$$

In other words, Corr_D *corepresents bivariant functors of* D.

Before proceeding to the argument (cf. 3.5.8), I list a few consequences of this representability statement.

3.5.2 Corollary (Unicity of correspondences)**.** *Let* $\mathrm{D} \to \mathrm{K}$ *be a 2-category of correspondences of* D. *There is a unique 2-equivalence* $\mathrm{K} \cong \mathrm{Corr}_\mathrm{D}$ *compatible with the inclusion of* D.

3.5.3 Corollary (Segal model)**.** *Any 2-category of correspondences is modelled by Gaitsgory-Rozenblyum's Segal object [GR17, Part V].*

Proof. By theorem 3.2.2 of *loc. cit*. the universal properties are the same. □

3.5.4 (Category of bivariant theories of D)**.** Write Biv_D for the full subcategory of $\mathrm{D} \downarrow 2\mathbf{Cat}$ spanned by the bivariant functors — that is, the Grothendieck integral of the 2-functor

$$\mathrm{K} : 2\mathbf{Cat} \underset{\mathbf{Cat}}{\vdash} \mathrm{Biv}(\mathrm{D}, \mathrm{K}) : 0\mathbf{Cat}.$$

By theorem 3.5.1, this is equivalent to the 2-functor $\mathrm{K}.2\mathrm{Fun}(\mathrm{Corr}_\mathrm{D}, \mathrm{K}) : 0\mathbf{Cat}$. (Here we only need the category of 2-functors to be 2-functorial as a map $2\mathbf{Cat}^{\mathrm{op}} \times 2\mathbf{Cat} \to 0\mathbf{Cat}$, which is established in [GR17, A.1].

3.5.5 Corollary (Universality)**.** *Restriction along* $\mathrm{D} \subseteq \mathrm{Corr}_\mathrm{D}$ *induces a 2-equivalence*

$$2\mathbf{Cat} \quad \vDash \quad \mathrm{Corr}_\mathrm{D} \downarrow 2\mathbf{Cat} \cong \mathrm{Biv}_\mathrm{D}.$$

That is, the correspondences theory is initial in the 2-category of bivariant theories of D.

3.5.6 Aside (3-categorical version). Clearly, Biv_D should most naturally be regarded as a 3-category; however, in the absence of a theory of path categories or Grothendieck integration for 3-categories it does not seem possible to make that statement.

3.5.7 (Naturality)**.** Correspondence categories are covariant for categories with fibre products and functors that preserve fibre products. More precisely, if we let

$$\mathbf{Biv} \subset 1\mathbf{Cat}^{\mathrm{fp}} \times_{2\mathbf{Cat}} 2\mathbf{Cat}^{\Delta^1}$$

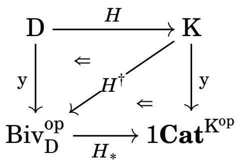
denote the full subcategory spanned by the bivariant functors — that is, the contravariant Grothendieck integral

$$\mathbf{Biv} := \int_{\mathrm{D}:1\mathbf{Cat}^{\mathrm{fp}}} \mathrm{Biv}_\mathrm{D}.$$

This is the global category of all bivariant functors.

By corollary 3.5.5, the full subategory of **Biv** spanned by the correspondence theories projects isomorphically onto $1\mathbf{Cat}^{\mathrm{fp}}$. This provides a section $1\mathbf{Cat}^{\mathrm{fp}} \to \mathbf{Biv}$ unique among those that send each D to a category of correspondences.

3.5.8 (Outline of proof)**.** Starting from a bivariant functor $H : \mathrm{D} \to \mathrm{K}$, we will construct a diagram



(9)

where $H^\dagger$ is obtained by restriction along H, and H_* by restriction along $H^\dagger$. More precisely, the induced functors $H^*, H_!$ are defined by formulas

$$\alpha : \mathrm{K},\ \beta : \mathrm{D} \qquad H^\dagger\alpha(\beta) = \mathrm{K}(\alpha, H\beta) \tag{10}$$

$$F : \mathrm{Biv}_{\mathrm{D}}^{\mathrm{op}},\ \alpha : \mathrm{K} \qquad H_*F(\alpha) = \mathrm{Biv}_{\mathrm{D}}(F, H^\dagger\alpha). \tag{11}$$

The composite 2-cell that intertwines the morphisms of the outer square can be identified with the *evaluation 2-cell* appearing in the bivariant Yoneda lemma 3.3.3. In other words, the outer square is commutative. In particular, H_* maps the Yoneda image of D into $\mathrm{K} \subset 1\mathbf{Cat}^{\mathrm{K}^{\mathrm{op}}}$.

The formation of this square is 2-natural in H, i.e. it defines a 2-functor of extension

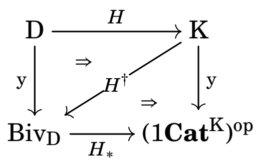
$$\mathrm{Biv}(\mathrm{D},\mathrm{K}) \underset{\mathbf{2Cat}}{\to} 2\mathrm{Fun}\left(\mathrm{Biv}_{\mathrm{D}}^{\mathrm{op}}, 1\mathbf{Cat}^{\mathrm{K}^{\mathrm{op}}}\right)$$

which, by the Yoneda lemmata, factors through the full subcategory of functors that send the objects of D into K. Composing with the restriction

$$2\mathrm{Fun}(\mathrm{D},\mathrm{K}) \times_{2\mathrm{Fun}\left(\mathrm{D},\mathbf{Cat}^{\mathrm{K}^{\mathrm{op}}}\right)} 2\mathrm{Fun}\left(\mathrm{Biv}_{\mathrm{D}}^{\mathrm{op}}, \mathbf{Cat}^{\mathrm{K}^{\mathrm{op}}}\right) \longrightarrow 2\mathrm{Fun}(\mathrm{Corr}_{\mathrm{D}}, \mathrm{K}),$$

we obtain an inverse to the restriction functor. Indeed, that this is a *right* inverse follows from the commutativity 2-cell of the square (9).

3.5.9 Aside (Opposite version). The same argument could be carried out equally with the covariant Yoneda mapping of D:



defined by

$$H^\dagger\alpha(\beta) := \mathrm{K}(H\beta, \alpha);$$

however, interpreting this formula as valued in $\mathrm{Biv}_{\mathrm{D}}$ necessitates passing through various op-formulas à la 3.1.7, and so in the interests of simplicity I opted for the former construction.

3.5.10 Aside (Natural version). It is somewhat perverse that this argument involves extending H to a morphism from a completion into a cocompletion. It is also possible to approach this using a completion (or cocompletion) on both sides, that is, replacing the bottom-left term by $(\mathbf{Cat}^{\mathbf{K}})^{\mathrm{op}}$. In this formulation, the bottom arrow would be a left adjoint

$$H_! : \mathrm{Biv}_{\mathrm{D}} \to \mathbf{Cat}^{\mathbf{K}^{\mathrm{op}}}$$

to the restriction functor.

A standard approach to constructing this diagram would be as follows:

i) The restriction functor into $\mathrm{Biv}_{\mathrm{D}} \subseteq \mathbf{Cat}^{\mathrm{D}}$ preserves 2-limits and 2-colimits, since this inclusion is closed under 2-limits and 2-colimits.

ii) By the adjoint 2-functor theorem and the fact that $\mathbf{Cat}^{\mathbf{K}}$ is 2-cocomplete, it admits a left adjoint.

iii) The commutativity of the outer square follows again from the bivariant and the 2-Yoneda lemmata.

Despite being more natural in the sense that it fits into a general covariance pattern of 2-presheaves, from the minimalist perspective of this paper it seems to unnecessarily entail several additional steps — particularly, an adjoint 2-functor theorem — hence I have not attempted to flesh out the full details.

3.5.11 (Yoneda adjoint)**.** The 2-functor of *Yoneda pullback*

$$(-)^\dagger : \mathrm{Biv}(\mathrm{D},\mathrm{K}) \to \mathrm{Fun}(\mathrm{K},\mathrm{Biv}_\mathrm{D}^{\mathrm{op}})^{\mathrm{op}}$$

with the formula (10) is obtained by repeated currying:

$$\begin{aligned}
& \underset{1}{H} : \mathrm{Biv}(\mathrm{D},\mathrm{K}),\ \underset{2}{\alpha} : \mathrm{K}^{\mathrm{op}},\ \underset{3}{\beta} : \mathrm{D}\,(\rightleftarrows) \quad \underset{\mathbf{2Cat}}{\vdash} \quad \mathrm{K}(\underset{2}{\alpha},\underset{1}{H}\underset{3}{\beta}) : \mathbf{1Cat} \\
\underset{\text{def}}{\equiv}\ & H : \mathrm{Biv}(\mathrm{D},\mathrm{K}),\ \alpha : \mathrm{K}^{\mathrm{op}} \quad \vdash \quad \beta.H^\dagger\alpha : \mathrm{Biv}_\mathrm{D} \\
\underset{\text{def}}{\equiv}\ & H : \mathrm{Biv}(\mathrm{D},\mathrm{K}) \quad \vdash \quad \alpha.\beta.H^\dagger : \mathrm{K}^{\mathrm{op}} \to \mathrm{Biv}_\mathrm{D}
\end{aligned}$$

and then applying the op-equivariance of 2Fun. Here on the second line I used the bivariant currying rule **??**.

The induced mapping on correspondences 3.3.4 defines a 2-cell $H^\dagger H \to \mathrm{y}_\mathrm{D}^{\mathrm{Biv}}$, to wit:

$$\begin{aligned}
& \alpha : \mathrm{D}^{\mathrm{op}},\ \beta : \mathrm{D}\,(\rightleftarrows) \quad \vdash \quad \mathrm{Corr}_\mathrm{D}(\alpha,\beta) \underset{\mathbf{1C}}{\to} \mathrm{K}(H\alpha,H\beta) \\
\underset{\text{def}}{\equiv}\ & \alpha : \mathrm{D}^{\mathrm{op}} \quad \vdash \quad H^\dagger H\alpha \underset{\mathrm{Biv}_\mathrm{D}}{\longrightarrow} \mathrm{y}_\mathrm{D}^{\mathrm{Biv}}\alpha \qquad \text{(substituting definitions)} \\
\underset{\text{def}}{\equiv}\ & 2\mathrm{Fun}\left(\alpha : \mathrm{D}, \mathrm{Biv}_{\beta:\mathrm{D}^{\mathrm{op}}}\right)\left(H^\dagger H, \mathrm{y}_\mathrm{D}^{\mathrm{Biv}}\right)
\end{aligned}$$

however, we will not make direct usage of this.

3.5.12 (Pushforward)**.** The second 2-functor (11)

$$2\mathrm{Fun}\left(\mathrm{K},\mathrm{Biv}_\mathrm{D}^{\mathrm{op}}\right)^{\mathrm{op}} \to 2\mathrm{Fun}\left(\mathrm{Biv}_\mathrm{D}^{\mathrm{op}}, \mathbf{1Cat}^{\mathrm{K}^{\mathrm{op}}}\right)$$

is obtained by the same logic as 3.5.11 applied to

$$\begin{aligned}
& F : \mathrm{Biv}_\mathrm{D}^{\mathrm{op}},\ G : \mathrm{K}^{\mathrm{op}} \to \mathrm{Biv}_\mathrm{D},\ \alpha : \mathrm{K}^{\mathrm{op}} \quad \underset{\mathbf{2Cat}}{\vdash} \quad \mathrm{Biv}_\mathrm{D}(F,G\alpha) : \mathbf{1Cat} \\
\underset{\text{def}}{\equiv}\ & F : \mathrm{Biv}_\mathrm{D}^{\mathrm{op}},\ G : \mathrm{K}^{\mathrm{op}} \to \mathrm{Biv}_\mathrm{D} \quad \vdash \quad \alpha.G^\dagger F : \mathrm{K}^{\mathrm{op}} \to \mathbf{1Cat} \\
\underset{\text{def}}{\equiv}\ & G : \mathrm{K}^{\mathrm{op}} \to \mathrm{Biv}_\mathrm{D} \quad \vdash \quad F.\alpha.G^\dagger : \mathrm{Biv}_\mathrm{D} \to \mathbf{1Cat}^{\mathrm{K}^{\mathrm{op}}}
\end{aligned}$$

The equation is (**??**). Again, there is a 2-cell $\mathrm{y}_\mathrm{K} \to G^\dagger G$; I omit the derivation. I write H_* for the composite $({}^\dagger H)^\dagger$.

Proof of theorem 3.5.1. The bivariant Yoneda lemma 3.3.3 provides us with a natural isomorphism between $\mathrm{Biv}(\mathrm{D},\mathrm{K}) \to \mathrm{Fun}(\mathrm{Corr}_\mathrm{D},\mathrm{K}) \to \mathrm{Biv}(\mathrm{D},\mathrm{K})$ and the identity.

Conversely, we have to compare

$$\begin{aligned}
& \alpha : \mathrm{K}^{\mathrm{op}},\ H : \mathrm{Corr}_\mathrm{D} \to \mathrm{K},\ x : \mathrm{Corr}_\mathrm{D} \quad \underset{\mathbf{2C}}{\vdash} \quad \mathrm{K}(\alpha,Hx) : \mathbf{1Cat} \\
\equiv\ & \alpha : \mathrm{K}^{\mathrm{op}},\ H : \mathrm{Corr}_\mathrm{D} \to \mathrm{K} \quad \vdash \quad x.({}^\dagger H)\alpha : \mathrm{Corr}_\mathrm{D} \to \mathbf{1Cat} \\
\equiv\ & H : \mathrm{Corr}_\mathrm{D} \to \mathrm{K} \quad \vdash \quad \alpha.x.({}^\dagger H) : \mathrm{K}^{\mathrm{op}} \to \mathbf{1Cat}^{\mathrm{Corr}_\mathrm{D}}
\end{aligned}$$

with

$$\begin{aligned}
\alpha : \mathrm{K}^{\mathrm{op}},\ H : \mathrm{Corr}_\mathrm{D} \to \mathrm{K},\ x : \mathrm{Corr}_\mathrm{D} \quad \underset{\mathbf{2Cat}}{\vdash} \quad & (H|_\mathrm{D})_* x(\alpha) : \mathbf{1Cat} \\
:= \quad & \mathrm{Biv}_\mathrm{D}(x,(H|_\mathrm{D})^\dagger\alpha) \qquad \text{def } 3.5.12
\end{aligned}$$

Note that by definition 3.5.11 $(H|_{\mathrm{D}})^\dagger \alpha = (H^\dagger \alpha)|_{\mathrm{D}}$.

We do this by lifting x and $(H|_{\mathrm{D}})^\dagger \alpha$ along the restriction $1\mathbf{Cat}^{\mathrm{Corr}_{\mathrm{D}}} \to \mathrm{Biv}_{\mathrm{D}}$, which fits into a commuting diagram of 2-categories

$$\begin{array}{ccc} & & (1\mathbf{Cat}^{\mathrm{Corr}_{\mathrm{D}}})^{\mathrm{op}} \\ & \overset{\mathrm{y}}{\nearrow} & \downarrow {\scriptstyle -|_{\mathrm{D}}} \\ \mathrm{Corr}_{\mathrm{D}} & \rightarrowtail & \mathrm{Biv}_{\mathrm{D}}^{\mathrm{op}} \end{array}$$

(a consequence of the 2-Yoneda lemma) so that it induces a 2-natural transformation

$$H : \mathrm{Corr}_{\mathrm{D}} \to \mathrm{K},\ \alpha : \mathrm{K}^{\mathrm{op}},\ x : \mathrm{Corr}_{\mathrm{D}} \quad \vdash \quad 1\mathbf{Cat}^{\mathrm{Corr}_{\mathrm{D}}}(\mathrm{y}x, H^\dagger \alpha) \to \mathrm{Biv}_{\mathrm{D}}(x, (H|_{\mathrm{D}})^\dagger \alpha)$$

which, coupled with the 2-categorical evaluation map for $H^\dagger \alpha$ and the bivariant Yoneda lemma for $(H|_{\mathrm{D}})^\dagger \alpha$, gives the result. □

4 Application: symmetric monoidal structures

Symmetric monoidal ($\mathbb{E}_\infty$) categories are defined and studied in chapter 2 of [HA]. Taking the definition there seriously, it is actually surprisingly difficult to define even the most basic symmetric monoidal structures, which are the *Cartesian* (§4.1 of *loc. cit.*) and *co-Cartesian* (§4.3) structures determined by *products* and *coproducts*, respectively. In fact, this issue was the original motivation for this note.

4.1 Co-Cartesian structure

4.1.1. Let **Fin** denote the category of finite sets. Segal's category Γ^{op} of finite pointed sets embeds in $\mathrm{Corr}_{\mathbf{Fin}}$

$$[n^+ \xrightarrow{f} m^+] \mapsto [n \leftarrow f^{-1}m \to m]$$

as the subcategory spanned by the correspondences whose contravariant part is injective. Let us write simply Corr for the 1-core of $\mathrm{Corr}_{\mathbf{Fin}}$.

A symmetric monoidal structure on a category C (def. 0.7 of *loc. cit.*) is the same data as a product-preserving 1-functor $\mathrm{C}^\otimes : \mathrm{Corr} \to 1\mathbf{Cat}$ together with an identification $\mathrm{C}^\otimes(1) \cong \mathbf{C}$. We will use the main result 3.5.1 to construct such functors.

4.1.2 (Coproduct operad)**.** Let C be a category, and let $\mathrm{C}^\sqcup$ be the full subcategory of its presheaf category whose objects are finite coproducts of representables. By passage to adjoints 3.2.1, the formation of $\mathrm{C}^\sqcup$ is 2-functorial in C.

Topos theory yields a natural forgetful functor

$$\pi_0 : \mathrm{C}^\sqcup \to \mathbf{Fin}.$$

Since coproducts in a topos are universal, this is a Cartesian fibration. The pullback functors along the n inclusions $1 \to n$ of sets identify $\mathrm{C}^\sqcup(n)$ with C^n.

This construction is natural in C, and sets up a fully faithful embedding

$$\mathrm{C} : 1\mathbf{Cat} \quad \underset{2\mathbf{Cat}}{\vdash} \quad \mathrm{C}^\sqcup : 1\mathrm{Cart}_{\mathbf{Fin}}$$

into the category of Cartesian fibrations over **Fin**. This is the 'coproduct operad'. The associated functor $\mathbf{Fin}^{\mathrm{op}} \to 1\mathbf{Cat}$ preserves products.

4.1.3 (Coproduct monoid)**.** Suppose now that C admits finite coproducts; that is, the inclusion $\mathrm{C} \to \mathrm{C}^\sqcup$ admits a left adjoint. Then the fibration $\mathrm{C}^\sqcup \to \mathbf{Fin}$ is also *co-Cartesian*: to a set map $\varphi : I \to J$, the pushforward functor is

$$(X_i)_{i:I} \mapsto \left(\sqcup_{\varphi i = j} X_i\right)_{j:J}.$$

The universal property of coproducts ensures that this map is indeed co-Cartesian over φ.

A functor $F : \mathrm{C} \to \mathrm{D}$ between categories with finite coproducts preserves these coproducts if and only if its extension $F^{\sqcup} : \mathrm{C}^{\sqcup} \to \mathrm{D}^{\sqcup}$ described in 4.1.2 preserves co-Cartesian morphisms; in other words, if it is a morphism of *bi-Cartesian* fibrations.

Hence, our construction yields a fully faithful functor from the category of categories admitting finite coproducts and finite-coproduct-preserving functors to CAlg(1**Cat**).

$$\mathrm{C} : 1\mathbf{Cat}^{\mathrm{fcop}} \quad \underset{2\mathbf{Cat}}{\vdash} \quad (\mathrm{C}, \sqcup) : \mathrm{CMon}(1\mathbf{Cat})$$

4.1.4 Aside. The analagous statements for categories with finite *products* and *Cartesian* symmetric monoidal structures follow, of course, immediately by passing to the opposite category.

All examples of symmetric monoidal structures (that I know of) are built from these by composing them with natural operations.

4.1.5 Aside. This description offers somewhat improved efficiency as compared with that of [HA, §2.4.1] — the proofs here are trivial, while those of *loc. cit.* (particularly proposition 1.6) occupy several pages. On the other hand, it seems to me that the former is conceptually close to that of [HA, §2.4.3], except that the latter takes place in the quasi-category model.

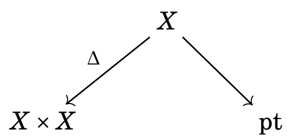
4.1.6 (Spans operad)**.** Suppose that D is finitely complete. By the argument of the preceding paragraph, $(\mathrm{D}, \times)$ is symmetric monoidal. (In this case $(\mathrm{D}, \times) = \mathrm{Corr}_{\mathrm{D}}(\mathrm{pt}, \mathrm{pt})$ as a symmetric monoidal category, so that $\mathrm{Corr}_{\mathrm{D}}$ provides a kind of delooping of D.) In fact, because products commute with other limits, it is even a commutative algebra in the category $1\mathbf{Cat}^{\mathrm{lex}}$ of finitely complete categories and left exact functors.

We may therefore compose

$$\mathrm{D}^{\times} : \mathbf{Fin} \to 1\mathbf{Cat}^{\mathrm{lex}}$$

with the functor $\mathrm{Corr} : 1\mathbf{Cat}^{\mathrm{lex}} \to 2\mathbf{Cat}$ of 3.5.7, and hence define a symmetric monoidal structure on $\mathrm{Corr}_{\mathrm{D}}$ compatible with the inclusion $(\mathrm{D}, \times) \to (\mathrm{Corr}_{\mathrm{D}}, \times)$. See also [Toë13, §2] for more details.[4]

This monoidal structure is discussed in [Hau14], where it is shown that all objects are actually *self-dual* with evaluation and coevaluation each given by the correspondence



considered in the appropriate direction.

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[4] Beware that despite the notation, this operation is *not* usually the product in $\mathrm{Corr}_{\mathrm{D}}$. Indeed, since $\mathrm{Corr}_{\mathrm{D}}$ is equivalent to its opposite, coproducts are products, and in typical examples where these exist (like **Fin**) it is given by the coproduct in the underlying category.

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