Foundations of Algebra A + B — lecture summary

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October 15, 2019

1 Algebra A

Lecture 1 Review of matrix linear algebra over a field $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$.

- Matrices $\mathrm{Mat}_{m\times n}(K)$, operation on column vectors and linearity. Basis vectors $e_i \in K^n$.
- Matrix multiplication, identity matrix, associativity (actually I didn't mention this but it is true and necessary to see that GL_n is a group). Multiplication corresponds to composition of linear maps.
- Inverse matrices and $\mathrm{GL}_n(K)$.
- Determinants, multiplicativity, and detecting invertibility with determinants.
- Abstract vector spaces and linear maps. Matrices represent linear maps on K^n.
- Bases and matrix representations of linear maps.
- Endomorphisms $\mathrm{End}(V)$ and $\mathrm{GL}(V)$. Linear groups.
- Examples. Special linear group. Shear transformations. Reflections.
- Examples. Commutative group of diagonal matrices (Cartan). Group of upper triangular matrices (Borel).

Lecture 2 Examples of linear groups, especially groups of upper triangular matrices.

- Multiplicative groups, groups of diagonal matrices (Cartan subgroup of GL_n, SL_n).
- Group of upper triangular matrices (Borel subgroup of GL_n).
- The Jordan canonical form theorem implies that every matrix over $\mathbb{C}$ is conjugate to an upper triangular matrix (that is every matrix belongs to a Borel subgroup). Moreover, a matrix belongs to a Cartan subgroup if and only if it is diagonalisable.
- Nilpotent matrices, unipotent matrices and their inverses (unipotent radical of Borel subgroup).
- Parametrising the Borel subgroup: $B \simeq \mathbb{G}_m^n \times \mathbb{G}_a^{\frac{1}{2}n(n-1)}$ as sets, but not as groups (with examples).
- Maximality of the Borel subgroup and Bruhat cells in the case of SL_2 — proofs deferred until next lecture.

2 Algebra B

Lecture 1 General background on sets and maps.

Lecture 2 Basic definitions and examples in group theory.

- Definition of groups. Uniqueness of identities and inverses, cancellative law.
- Fundamental examples: permutation group, general linear group, automorphisms of a 'mathematical object'.
- Abelian groups, plus fundamental examples (additive groups, vector groups, multiplicative groups).
- Group homomorphisms. The inverse of a bijective group homomorphism is a homomorphism. Isomorphisms.
- Sets of group homomorphisms, group structure in Abelian case. Group of group automorphisms.
- Definition of group action on a set.
- Subgroups, subgroup generated by a subset.

Lecture 3 Kernels of homomorphisms, various examples.

- Cyclic groups, classification (statement only). Orders of elements, orders of finite groups.
- Circle group, realisation of cyclic groups as complex roots of unity.
- Kernel of a group homomorphism. Reformulation as a short exact sequence (SES). A homomorphism with trivial kernel is injective.
- Opposite group and inverse homomorphism $G \cong G^{\mathrm{op}}$.
- Cartesian product group. Example: $\mathbb{C}^\times \cong \mathbb{R} \times S^1$.
- Exponential sequence $\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times$.
- Free groups (especially the free group $\mathbb{Z}$ on one generator).
- Translation action of a group on itself (by permutations).
- Adjoint action of a group on itself (by group automorphisms). Inner automorphisms and centre.

Definitions and existence of quotients.

- G-equivariant maps and quotients of G-sets. Left and right G-actions.
- Faithful and transitive G-actions.
- Stabiliser groups. The stabiliser group of a translate of x is related by conjugation to $\mathrm{Stab}(x)$.
- Quotient of a group by a subgroup as a transitive G-set with specified stabiliser.
- Normal subgroups. The kernel of a group homomorphism is normal.